Regime Switching Rough Heston Model

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Abstract

The regime switching rough Heston model has two important features on different time scales. The regime switching is motivated by changes in the long term behaviour. The parameter of the model might change over time due to macro-economic reasons. Therefore we introduce a Markov chain to model the switches in the long term mean of the volatility. The rough behaviour is a more local property and is motivated by the stylized fact that volatility is less regular than a standard Brownian motion. Therefore the driving noise in the model is a fractional Brownian motion. We derive and implement pricing formulae for call and put option and then add some insights into the effects of the rough behaviour and the regime switches to these prices. The techniques are much more involved than for the standard Heston model, since the rough processes do neither have the Markov property nor the semi-martingale property. The regime switches introduce as an additional complexity time inhomegeneity.

Key words: Rough Browian Motion, Regime Switching, Heston Model, Analytic Pricing Formula, Full and partial Monte-Carlo-Methods

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1 Introduction

The most celebrated and widely used stochastic volatility model is the model by Heston (1993). In that model the asset price S follows a geometric Brownian motion and the stochastic volatility follows a square-root-process, also known as CIR-process which was pioneered by Cox et al. (1985). The dynamics of Heston model under the risk-neutral probability measure $\mathbb{Q}$ is given by:

$$\begin{aligned} dS_t &= rS_t dt + S_t\sqrt{V_t}dB_t \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t, \end{aligned} \tag{1}$$

with two possible correlated Brownian motions B and W. One important advantage of this stochastic volatility model is its analytic tractability. It enables the modeller to infer the parameters of the process from the market quoted option prices fairly easily.

However, in a recent paper by Gatheral et al. (2014), it is shown that time series of realized volatility are rough with a Hurst parameter H less than one-half, in particular near zero or of 0.1. In addition in Jaisson and Rosenbaum (2016) and Euch and Rosenbaum (2017) a micro-structure market model is based on self-exciting Poisson process, so called Hawkes processes, which converge to rough Brownian motions.

In the rough Heston model, the Brownian motion W driving the volatility (in a sense of classical approach) is now replaced by a rough (fractional-) Brownian motion $W^H, H \in (0, 0.5)$. Equation (1) can be re-written (informally) in a fractional stochastic volatility framework as introduced by Comte and Renault (1998) as follows:

$$
\begin{aligned}
dS_t &= rS_t dt + S_t\sqrt{V_t}dB_t \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^H. \qquad (2)
\end{aligned}
$$

Another stream of research, as described in Elliott et al. (2005) and Elliott et al. (2016) argues that asset prices or the associated volatility process should exhibit changing regime. They referred to example on the statistical analysis by Maghrebi et al. (2014), that the model should have at least two regimes under the risk neutral measure. Also several papers (Hamilton and Susmel, 1994; Moore and Wang, 2007; So et al., 1998) showed that index volatilities are subjected to regime switches under the physical measure.

The economic consideration is one important motivation to use regime switches using Markov chains instead of jump-diffusion in order to incorporate sudden changes in volatility. Also a combination of rough Brownian motion and jump processes seems not to be considered in the literature as of yet. We restrict ourselves to changes in the mean-reversion parameter since that model maintains the analytic tractability. As a consequence many option pricing formulae can be obtained at least in a semi-analytic form. Another argument for regime switching models are that those are used for pricing in many other cases, like Overbeck and Weckend (2017), Yuen and Yang (2010), Alexander and Kaeck (2008), and Ang and Bekaert (2002). Calibration of the regime switching models have been analysed in Mitra (2009) and He and Zhu (2017). In the case of rough Heston model calibration is still a major open problem, since even employing semi-analytic solutions is a computationally expensive exercise. For early exploration of calibration of rough Heston model see a technical report by Alfeus et al. (2017).

Since stochastic volatility models are usually not complete, there are several equivalent martingale measure. It is widely accepted that as long as volatility is not traded, the so-called minimal martingale measure will not change the volatility process and therefore regime-switching will be also passed on to the pricing measure.

Our paper will now introduce a new Heston type model which covers two important generalization of the classical model, namely the rough volatility model and regime switching volatility. The so-called rough regime switching Heston model will inherit the analytic tractability of the rough Heston model, which was derived in Euch and Rosenbaum (2016, 2017) and the tractability of the regime switching extension as in Elliott et al. (2016). Two important stylized features of volatility, namely the rough behaviour in its local behaviour, and the regime switching property consistent with more long term economic consideration can be accommodated in one consistent model approach.

In the classical Heston model the Laplace-transform of the log asset price is a solution to a Riccati-equation. Although this result require the semi-martingale and Markov-property of the asset and volatility process, a totally analogous result can be proved for the rough Heston model, where the volatility is neither a semi-martingale nor a Markov process. The Riccati equation, which is an ordinary differential equation is now replace by a rough integral equation, see Euch and Rosenbaum (2016). Moreover this results is extended to a time dependent long term mean reversion level $\theta_s, s \in [0, T]$. Exactly in this formula time dependent θ is required in order to extend the resolvent equation as in Elliott et al. (2005) and Elliott et al. (2016) to our case. However, in our setting the resolvent equation, which is an equation associated with the Markovian regime switching process for θ_s, depends also on the final time T. This dependency increases considerably the mathematical and numerical complexity. From the computational point of view, the model is computationally challenging as firstly we have to compute the matrix ODE (as proposed by Elliott and Nishide (2014)) using Runge–Kutta method and secondly we have to compute the fractional differential Riccati equation using predictor–corrector schemes in lines of Adam's method (see for instance

Diethelm et al. (2002)). Generally, merging the two modeling frameworks is slower mainly due to the time consuming resolvent equation in the matrix ODE and the predictor–collector schemes. A fast method to compute fractional differential Riccati equation is not yet known.

In the paper we are able to extend the arguments from Euch and Rosenbaum (2016) as well as from Elliott et al. (2016) to finally derive an analytic representation of the Laplace-functional of the asset price. By Fourier-inversion technique analytic pricing formulae for put and calls are given.

We benchmark these semi-analytic prices against two types of Monte-Carlo-simulations. One is a full Monte-Carlo simulation, in which the three dimensional stochastic processes (B, W, θ) is simulated and the option payout can be obtained (in the risk neutral world) in each simulation.

The second, a novel method in this context, is the partial Monte-Carlo-Simulation. Here we simulate the path of $\theta_s(\omega), s \in [0, T]$ and then solve the corresponding rough Riccati equation. Here we are able to avoid the resolvent equation which was shown to be very time consuming. This computation method is an innovation compared to the approaches presented in the literature and is shown to be the most effective one.

Despite the different computation time, the results of the three methods are very close. This is in contrast to the results of Elliott et al. (2016), in which however only the two classical computation methods were considered. For reason not explained they only considered maturities upto year 1 and it is apparent that the difference between MC and analytic increases with maturity. This we can not observe. As a test we run the Monte-Carlo simulation without changing the regime, i.e. do not simulate the Markov chain of regime switches. Then our results are closer to the Monte-Carlo based figures reported by Elliott et al. (2016) in their numerical results (see Table 4).

In the section numerical results, we present the three different calculation methods. In addition we show that the call option price as a function of the

Hurst parameter can exhibit different shapes. We see in our example that for shorter maturities call prices are increasing with increasing Hurst parameter, i.e. rough prices based on rough volatility are cheaper than those based Brownian motion prices and prices based on long memory volatility are even more expensive than Brownian motion. This changes if maturity increases. At a certain level Brownian volatility prices are the most expensive one and both rough and long term volatility based prices are lower (see Figure 1).

We also analyse the sensitivity with respect to average number of regime until maturity, with respect to initial volatility and the correlation between W^H and B.

The paper is structured as follows. Section 2 introduces the model and pricing methodologies. Section 3 presents numerical results and call price sensitivity with respect to Hurst parameter. Section 4 concludes.

2 Basic Model Description

We directly work under the pricing measure for the underlying (already discounted) asset S. From Equation (2) the log prices $X = \log S$ then become

$$\begin{aligned} dX_t &= (r - V_t/2)dt + \sqrt{V_t}dB_t \\ V_t &= V_0 + \frac{\kappa}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta - V_s)dt + \frac{\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dW_s \end{aligned} \quad (3)$$

We now incorporate a regime switching into the mean reversion level θ as in Elliott et al. (2016) and choose the rough volatility model by Euch and Rosenbaum (2016). This leads to the following stochastic integral equation for V

$$V_t - V_0 = \frac{\kappa}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta_s - V_s)dt + \frac{\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dW_s, \quad (4)$$

where W is now a standard Brownian motion having correlation ρ with B and $(\theta_s)_{0\leq t<\infty}$ is a finite state time homogeneous Markov process with generator matrix Q independent of S and W.

2.1 Fixed function $s \to \theta_s$

We need the following result from Euch and Rosenbaum (2017) that for a fixed function $s \to \theta_s$ the characteristic function of $X_t = \log S_t$ equals

$$E[e^{zX_t}] = \exp\left(\int_0^t h(z,t-s)\left(\kappa\theta_s + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)}\right)ds\right), z \in \mathbb{C}, \quad (5)$$

where h is the unique solution of the following fractional Riccati equation:

$$\begin{aligned} D^\alpha h &= \frac{1}{2}(z^2 - z) + (z\rho\sigma - \kappa)h(z,s) + \frac{\sigma^2}{2}h^2(z,s), s < t, z \in \mathbb{C}, \quad (6) \\ I^{1-\alpha}h(z,0) &= 0. \end{aligned}$$

Here the fractional differentiation and integral are defined by

$$D^\alpha h(z,s) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} h(z,s)ds \quad (7)$$

$$I^\alpha h(z,s) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(z,s)ds \quad (8)$$

2.2 Regime switching θ_s

As in Elliott et al. (2016) we define $\theta_s(\omega) = \sum_{i=1}^k \vartheta_i Z_s^{(i)}(\omega) = \langle \boldsymbol{\vartheta}, Z_s\rangle$ where Z is a Markov chain, independent from (S, V) with state space the set of unit vectors in $\mathbf{R}^k$, i.e.$Z_s \in \{e_i = (0,..,1,0..)^T, i = 1,..,k\}$ and ϑ is the vector of k-different mean reversion levels. The infinitesimal generator of the process Z is also denoted by Q i.e. q_{ij} is the intensity of switching from state e_i to e_j, i.e. for θ itself the intensity of switching from ϑ_i to ϑ_j.

The following proposition is the main new mathematical results of the paper and lays the foundation of the numerical calculation of the option prices as presented below.

Proposition 2.1. *The conditional characteristic function of the random variable X with fixed T is given by:*

$$E[e^{zX_T}] = E\left[\exp\left(\kappa\int_0^T h(z,T-s)\langle\boldsymbol{\vartheta}, Z_s\rangle ds\right)\right] e^{\int_0^T h(z,T-s)\frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)}ds}. \quad (9)$$

Proof. Fix the final time T and consider now the processes

$$g_t = \exp\left(\kappa\int_0^t h(z,T-s)\langle\boldsymbol{\vartheta}, Z_s\rangle ds\right) \quad (10)$$

$$G_t = g_t Z_t \quad (11)$$

We have that

$$\begin{aligned} dG_t &= g_t dZ_t + Z_t dg_t \quad (12)\\ &= g_t(Q' Z_t dt + dM_t^Z) + Z_t g_t h(z,T-t)\langle\boldsymbol{\vartheta}, Z_t\rangle dt \end{aligned}$$

and can proceed exactly as in Elliott et al. (2016). Therefore

$$\begin{aligned} dG_t &= \left(Q' + \kappa h(z,t)\langle\boldsymbol{\vartheta}, Z_t\rangle\right) g_t Z_t dt + g_t dM_t^Z \quad (13)\\ &= \left(Q' + \kappa h(z,T-t)\Theta\right) g_t Z_t dt + g_t dM_t^Z \end{aligned}$$

Once this is done we will finally end up with a matrix ODE as in Elliott et al. (2016), i.e.

$$\frac{d\Phi(u,t)}{dt} = \left(Q' + \kappa h(z,T-t)\Theta\right)\Phi(u,t), \; u < t, \; \text{ with } \; \Phi(u,u) = \mathbf{I}. \quad (14)$$

We now get that

$$E\left[G_t\right] = \Phi(0,t)Z_0, \tag{15}$$

and because $\forall t, \quad \langle Z_t, \mathbf{1}\rangle = 1$, we have

$$E\left[\exp\left(\kappa\int_0^T h(z,s)\langle\boldsymbol{\vartheta}, Z_s\rangle ds\right)\right] = \langle\Phi(0,T)Z_0, \mathbf{1}\rangle. \tag{16}$$

In summary, combining Equations (8), (9), and (16) the regime switching rough Heston model has the characteristic representation given by:

$$\varphi_X(z) = E[e^{zX_T}] = \exp\left(V_0 I^{1-\alpha}h(z, T-\cdot)\right)\langle\Phi(0,T)Z_0, \mathbf{1}\rangle, \tag{17}$$

where

$$I^{1-\alpha}h(z,T-\cdot) = \int_0^T h(z,T-s)\frac{s^{-\alpha}}{\Gamma(1-\alpha)}ds. \tag{18}$$

$\square$

This characteristic function in Equation 9 is used in the semi-analytic pricing method below.

2.3 Monte-Carlo Simulation

As benchmark for the semi-analytic pricing method via the characteristic function which involves the rough Riccati equation and the matrix equation, we develop Monte-Carlo simulation based approaches, i.e., we carry out two types of Monte-Carlo simulation. In the first one only the regime switching process is simulated and for each path of θ the corresponding Laplace-functional is calculated. In that way the performance of the ordinary matrix differential equation is tested against Monte-Carlo simulation. The second one is a straightforward simulation of the three dimensional process (θ, V, S).

2.3.1 Partial Monte-Carlo

We simulate the paths of θ_s and then evaluate for each realization $\theta_s(\omega)$, the formula (5). A path $\theta(\omega)$ has the form

$$\theta_s(\omega) \quad = \quad \sum_{i=1}^{\infty} \mathbf{1}_{[S_{i-1}(\omega),S_i(\omega)[}(s) X_i(\omega), \tag{19}$$

where $S_0 = 0$, $S_i(\omega) = S_{i-1}(\omega) + T_i(\omega)$, where T_i, X_i are successively drawn from an exponential distribution with parameter $-q_{X_{i-1}(\omega)X_{i-1}(\omega)}$ and X from the jump distribtuion of Q i.e.

$$T_i \quad \sim \quad \exp(-q_{X_{i-1}(\omega)X_{i-1}(\omega)}) \tag{20}$$

$$P[X_i = \theta_k | X_{i-1}] \quad = \quad \frac{q_{X_{i-1}(\omega)k}}{-q_{X_{i-1}(\omega)X_{i-1}.(\omega)}} \tag{21}$$

Let us generate N of those paths $\theta(\omega_l), l = 1, .., N$ and evaluate for each $\theta(\omega_l)$ the expression

$$E[e^{zX_t}](\omega_l) \quad := \quad \exp\left(\int_0^t h(z, t-s) \left(\kappa\theta_s((\omega_l)) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)}\right) ds\right), \tag{22}$$

then

$$E[e^{zX_t}] \quad \sim \quad \frac{1}{N} \sum_{l=1}^{N} E[e^{zX_t}](\omega_l) \tag{23}$$

2.3.2 Full Monte-Carlo

Here we want to calculate the option price directly be Monte-Carlo simulation. We first simulate the 3-dimensional process (B, W^H, θ). From the $\theta_s(\omega)$ simulated as above, we build the values of the regimes at each of the discrete time steps t_i, at which we also want to generate the values of the volatility V_{t_i}, which depends on θ_{t_i} and the asset price S_{t_i}.

The Heston model itself is then defined via an Euler scheme according to (2), and option prices are obtained by evaluating the payoff at each path and taking the average over all Monte Carlo paths.

2.4 Analytic pricing based on Fourier transformation

To price options, we use the well-known Fourier-inversion formula of Gil-Pelaez (1951) (for convergence analysis see Wendel (1961)) which leads to a semi-analytic closed-form solution given by:

$$C_0 = e^{-rT}\mathbb{E}\left[(e^X - K)^+\right] = \mathbb{E}[e^X]\Pi_1 - e^{-rT}K\Pi_2, \tag{24}$$

where the probability quantities Π_1 and Π_2 are given by:

$$\begin{aligned}
\Pi_1 &= \mathbb{E}[e^X \mathbb{I}_{\{e^X>K\}}]/\mathbb{E}[e^X] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-\mathrm{i}z\log(K)}\varphi_X(z-i)}{\mathrm{i}z\varphi_X(-\mathrm{i})}dz\right] \qquad (25)\\
\Pi_2 &= \mathbb{P}\{e^X > K\} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-\mathrm{i}z\log(K)}\varphi_X(z)}{\mathrm{i}z}dz\right].
\end{aligned}$$

3 Numerical results

Most of the model parameters are adopted from Elliott et al. (2016), see Table 1. For the roughness case, we chose the Hurst parameter $H = 0.1$, as indicated by Gatheral et al. (2014) (see also Alfeus et al. (2017) for Hurst parameter estimation from realized variance and the calibration of rough Heston model).

Our first analysis begins with the test of the observation in Elliott et al. (2016) that with longer time to maturity Monte-Carlo prices diverge considerable from analytic prices. This we can not confirm. In percentage of price the Monte-Carlo error only increases slightly. These results are displayed in Table 2–3. However if we do not simulate the regime switches in the Monte Carlo simulation we observe the same increase as reported in Elliott et al. (2016), see Table 4 and Table 6. In our implementation we could neither observe the problems with maturity larger than 1 year nor the problems with the discontinouities in the complex plane as reported in Elliott et al. (2016). However, we can only observe that semi-analytic pricing suffers for the out of the money options.

In the second analysis we show how the Hurst parameter impacts the option price. Surprisingly this depends on the maturity of the option. We show the result without regime switching in the Figure 1. We get increasing, hump and then decreasing shapes for time to expiry bigger than 1.85 years.

Thirdly, we report on call prices under rough volatility (with $H = 0.1$) with different volatility and correlation assumptions, see Tables 5–7. Here we also exhibit the partial Monte-Carlo results. The prices are close to full Monte-Carlo, but it is faster, and sometimes even closer to semi-analytic. Approximately 1 000 000 simulations of the regime switches consume the same computation time as the semi-analytic calculations. Different to the Q_E-matrix used by Elliott et al. (2016) which roughly allows for one change per year, we also consider the case with multiple switches per year. The differences to the non-regime switching becomes larger, e.g. for $K = 100, T = 1$, we have 8.28 without regime switching, 9.4 with few and 10.14 with more regime switches, see Tables 5, 7 and Table 10.. Also the impact of the correlation assumption is dominant, and the starting value of the volatility has large influence. These numerical results are given in Tables 8–9.

Our last figure shows the implied volatility surface for different Hurst parameter, but with two different starting volatility but the same regime switching parameters and maturity $T = 1$. The lowest and steepest is the most rough one in the lowest regime, see Figure 2. This is naturally since option prices are cheaper under those parameters. The last Table 11 compares the speed across different computational methods.

Table 1: Model parameters

Parameters	value
$S(0)$	100
r	0.05
σ	0.4
ρ	-0.5
κ	3
$\theta_0 = [\theta^1\ \theta^2]$	[0.025 0.075]
α	1 ($\sim H = 0.5$)
Q_E	$\begin{bmatrix} -1 & 1 \\ 0.5 & -0.5 \end{bmatrix}$
No. of Simulations	1 000 000
Time Steps	250

Table 2: Call prices, $v_0 = 0.02 < \theta^1 < \theta^2$

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	**0.25**			**0.5**			**0.75**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	11.57148	0.00745	11.48849	13.27125	0.01006	13.25924	15.49823	0.01392	15.01218
95	7.38753	0.00648	7.17254	9.59107	0.00929	9.27515	10.73807	0.01015	11.24349
100	3.97505	0.00504	3.62547	6.68191	0.00842	5.88019	7.39669	0.00878	7.95919
105	1.69219	0.00338	1.33534	3.66323	0.00604	3.29490	4.71588	0.00721	5.28047
110	0.55198	0.00192	0.35727	1.83362	0.00427	1.62772	2.73756	0.00558	3.27974
115	0.14608	0.00098	0.08044	0.63928	0.00244	0.73575	1.71990	0.00460	1.92863
120	0.03219	0.00045	0.01697	0.35077	0.00187	0.31933	1.76078	0.00526	1.09545

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**0.25**			**0.5**			**0.75**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	11.57717	0.00729	11.73234	13.71997	0.01122	13.87580	15.72714	0.01448	15.84949
95	7.36691	0.00633	7.63948	9.94275	0.00999	10.17181	12.48412	0.01384	12.34856
100	3.90727	0.00489	4.31660	6.73111	0.00851	7.02647	9.49103	0.01238	9.29809
105	1.60708	0.00323	2.02691	4.20529	0.00688	4.53709	6.93522	0.01078	6.74947
110	0.49681	0.00179	0.77241	2.74969	0.00587	2.72878	4.87362	0.00915	4.71991
115	0.12359	0.00089	0.24232	1.06738	0.00340	1.53220	3.22032	0.00740	3.18403
120	0.02729	0.00041	0.06548	0.60658	0.00264	0.80856	2.16359	0.00611	2.07808

Table 3: Call prices, $v_0 = 0.02 < \theta^1 < \theta^2$, with longer maturities

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	**1**			**3**			**5**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	17.75749	0.01779	16.70273	29.72727	0.03784	27.80856	36.31831	0.04548	35.97219
95	12.00087	0.01112	13.09527	26.98125	0.03669	24.85439	33.51644	0.04393	33.34277
100	8.64760	0.00979	9.90182	22.91025	0.03127	22.10464	30.63824	0.04147	30.85462
105	6.92472	0.01005	7.19639	15.80680	0.01917	19.56451	27.16105	0.03704	28.50740
110	5.29075	0.00943	5.02534	13.21306	0.01786	17.23587	27.83889	0.04427	26.29963

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**1**			**3**			**5**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	18.04682	0.01869	17.64130	27.29340	0.03006	28.54621	35.22786	0.04123	36.00426
95	14.73782	0.01734	14.27432	24.46840	0.02925	25.69094	32.42804	0.03981	33.37863
100	11.84495	0.01585	11.29129	24.57055	0.03566	23.03282	33.21863	0.05016	30.89413
105	9.30708	0.01429	8.72376	21.82207	0.03334	20.57358	31.56898	0.05076	28.55036
110	7.19260	0.01271	6.58292	19.96980	0.03286	18.31214	27.14623	0.04221	26.34577

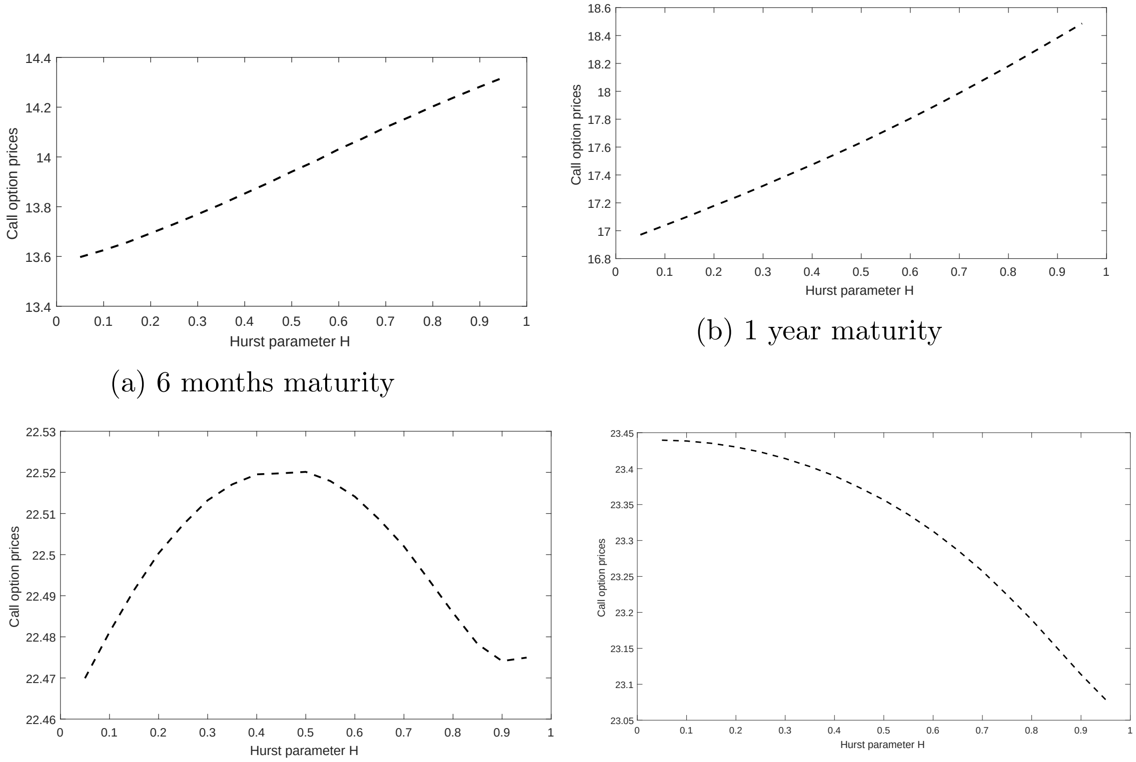
Table 4: Call prices, $v_0 = 0.02 < \theta^1 < \theta^2$, without simulation of regime switches in Monte Carlo

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	0.25			0.5			1		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	11.31855	0.00036	11.47998	12.78403	0.00080	13.26191	15.54240	0.00135	16.70267
95	6.91745	0.00049	7.17261	8.68745	0.00097	9.27230	11.74966	0.00158	13.09533
100	3.45268	0.00044	3.63632	5.34249	0.00100	5.88245	8.50439	0.00174	9.90178
105	1.40765	0.00048	1.31659	2.99190	0.00103	3.29437	5.91387	0.00183	7.19640
110	0.51416	0.00051	0.37300	1.57963	0.00104	1.62594	3.97996	0.00183	5.02538
115	0.17973	0.00037	0.08148	0.81441	0.00097	0.73872	2.62430	0.00177	3.38626
120	0.06229	0.00021	0.03019	0.42068	0.00080	0.31788	1.70823	0.00164	2.22082

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**0.25**			**0.5**			**1**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
90	11.60323	0.00033	11.73305	13.75855	0.00061	13.87595	17.77453	0.00096	17.64129
95	7.53515	0.00034	7.63789	10.16683	0.00062	10.17166	14.58900	0.00102	14.27432
100	4.35194	0.00031	4.31886	7.20199	0.00063	7.02656	11.81142	0.00106	11.29129
105	2.23511	0.00033	2.02486	4.90186	0.00064	4.53710	9.44866	0.00110	8.72376
110	1.04208	0.00037	0.77292	3.22584	0.00067	2.72865	7.47633	0.00112	6.58292
115	0.45445	0.00035	0.24399	2.06709	0.00069	1.53236	5.86090	0.00112	4.85611
120	0.19003	0.00027	0.06291	1.29924	0.00068	0.80852	4.56132	0.00113	3.50796



(a) 6 months maturity

(b) 1 year maturity

(c) 1.85 years maturity

(d) 2.5 years maturity

Figure 1: The impact of Hurst parameter on option values with changing expiry time

Under the rough case, we consider a case when $H = 0.1$ as empirically proven by Gatheral et al. (2014). At moment we are considering the generator matrix Q_E given above.

Table 5: Call prices under rough volatility, $v_0 = 0.02 < \theta^1 < \theta^2$ and Q_E generator

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	**0.25**				**0.5**				**1**			
	Monte Carlo			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**
	Full	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**
80	21.21834	0.08142	21.04928	20.99613	22.70236	0.10612	22.39585	21.92995	24.95195	0.14809	24.43662	24.54135
85	16.53185	0.07505	16.79874	16.89825	18.25055	0.10175	18.15340	17.71934	20.87358	0.14293	20.32966	20.28785
90	11.98653	0.06997	12.31647	11.94876	13.60705	0.09478	13.86663	13.37147	16.74382	0.13051	16.48881	16.32680
95	7.73380	0.06096	7.62077	6.67428	9.73972	0.08340	9.91929	8.89896	12.74631	0.11821	12.76987	12.64163
100	3.81818	0.04910	3.78302	2.46191	6.02765	0.07292	6.42048	4.93519	9.40263	0.11208	9.42984	9.30234
105	1.44438	0.03752	1.51491	0.24732	2.96650	0.05546	3.80160	2.15772	6.43873	0.09919	6.60637	6.46152
110	0.42772	0.02180	0.39556	0.12512	1.40261	0.04262	2.06220	0.75938	3.86032	0.08174	4.35670	4.24546

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**0.25**				**0.5**				**1**			
	Monte Carlo			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**
	Full	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**
80	21.22925	0.07501	20.98623	20.96890	22.79630	0.11497	22.15238	22.14655	25.74824	0.05678	25.50586	24.97705
85	16.52081	0.07119	16.69764	16.75793	18.18483	0.10556	17.86022	17.80336	21.70714	0.05376	21.55210	20.86047
90	11.96638	0.06593	12.11152	12.04218	13.99007	0.09802	13.93820	13.62524	17.96501	0.05086	17.89002	17.03892
95	7.50552	0.05791	7.50617	7.16747	9.84658	0.08937	10.07705	9.61300	14.15027	0.04681	14.58459	13.55432
100	3.63218	0.04688	3.79966	3.17796	6.30056	0.07959	6.45585	6.06377	11.00267	0.04286	11.63212	10.45512
105	1.19131	0.03641	1.33402	0.84507	3.44352	0.06389	3.79153	3.35864	8.10423	0.03822	9.08070	7.79629
110	0.38912	0.02242	0.51241	0.12238	1.70127	0.05041	2.05766	1.65555	5.76181	0.03393	6.93307	5.61828

Table 6: Call prices, $v_0 = 0.02 < \theta^1 < \theta^2$, without simulation of regime switches in Monte Carlo; rough case

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	0.25			0.5			1		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
80	21.19948	0.06566	20.99613	22.47280	0.08855	21.92995	24.87668	0.12432	24.54135
85	16.50078	0.06101	16.89825	17.72256	0.08502	17.71934	20.64484	0.11906	20.28785
90	11.68174	0.05562	11.94876	13.28680	0.07711	13.37147	15.91675	0.10802	16.32680
95	7.14933	0.04882	6.67428	8.90970	0.06915	8.89896	12.01290	0.10022	12.64163
100	3.10499	0.04128	2.46191	5.02244	0.05764	4.93519	8.28652	0.08782	9.30234
105	0.77033	0.02745	0.24732	2.24329	0.04713	2.15772	5.03983	0.07613	6.46152
110	0.26740	0.01873	0.12512	0.83912	0.03299	0.75938	2.88119	0.06492	4.24546

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**0.25**			**0.5**			**1**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
80	21.55874	0.09134	20.96890	23.03974	0.12825	22.14655	26.12745	0.18946	24.97705
85	16.61640	0.08343	16.75793	18.68628	0.12284	17.80336	22.12496	0.18115	20.86047
90	12.20763	0.07790	12.04218	14.49531	0.11147	13.62524	17.86913	0.16392	17.03892
95	7.84079	0.06585	7.16747	10.37963	0.10010	9.61300	14.37190	0.15299	13.55432
100	4.31448	0.05577	3.17796	6.90396	0.08765	6.06377	11.37817	0.13983	10.45512
105	1.82449	0.04377	0.84507	4.19076	0.07165	3.35864	8.53524	0.12659	7.79629
110	0.66541	0.02951	0.12238	2.37847	0.05910	1.65555	6.24291	0.11296	5.61828

In what follows, we consider a generator of the Markov chain

$$Q = \begin{bmatrix} -5 & 5 \\ 4 & -4 \end{bmatrix}.$$

Meaning, we are considering 5 jump rate per year from state 1 to state 2 and 4 jump rate from state 2 to state 1.

Table 7: Call prices under rough volatility and $v_0 = 0.02 < \theta^1 < \theta^2$

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	**0.25**				**0.5**				**1**			
	Monte Carlo			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**
	Full	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**
80	21.47068	0.07908	21.00175	20.98273	22.63093	0.10858	22.28225	22.01076	25.42472	0.15965	25.79463	24.70909
85	16.66761	0.07313	16.70332	16.84830	17.97404	0.10169	18.06484	17.72671	21.14268	0.14998	20.03459	20.48775
90	11.80479	0.06609	12.27504	11.98113	13.90196	0.09522	13.89394	13.45825	17.08855	0.14001	16.49172	16.56613
95	7.59101	0.05911	7.61382	6.84312	9.68327	0.08705	9.99224	9.19971	13.44172	0.13147	13.35790	12.97795
100	3.77509	0.04935	3.78298	2.70477	6.05559	0.07620	6.41210	5.42301	10.14287	0.12180	10.23271	9.77358
105	1.24446	0.03634	1.39731	0.44667	3.19847	0.06080	3.79761	2.66488	6.97217	0.10751	7.34930	7.03425
110	0.46183	0.02466	0.50095	0.04683	1.50331	0.04772	2.08867	1.11439	4.69907	0.08864	5.18584	4.82989

(b) Stating in a high state: $\theta_0 = \theta^2$

K/T	**0.25**				**0.5**				**1**			
	Monte Carlo			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**	**Monte Carlo**			**Semi-Analytic**
	Full	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**	**Full**	**std Error**	**Partial**	**Fourier**
80	21.27567	0.07903	21.07213	20.97415	22.70422	0.11488	22.25050	22.04900	25.71770	0.16603	24.45007	24.74451
85	16.51986	0.07355	16.78900	16.79487	18.38982	0.10809	17.98040	17.73785	21.25906	0.15787	21.93313	20.53618
90	12.02166	0.06611	12.01867	12.00798	13.84454	0.09795	13.84289	13.49585	17.24327	0.14703	17.88039	16.62739
95	7.57394	0.05978	7.72731	7.01520	9.88294	0.08755	9.86080	9.31996	13.34188	0.13406	13.78744	13.05632
100	3.69629	0.04933	3.79923	2.96032	6.05455	0.07464	6.43077	5.62394	10.13329	0.12298	10.44554	9.87387
105	1.31239	0.03849	1.47665	0.66300	3.50841	0.06450	3.87000	2.88394	7.42620	0.10936	7.83990	7.15294
110	0.53143	0.02878	0.42124	0.04452	1.72727	0.04967	2.07582	1.28172	5.01555	0.09459	5.83673	4.95390

Table 8: Call prices under regime-changing and rough volatility

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	**0.5**			**2**			**3**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Full	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**
90	14.09136	13.92905	13.44292	24.45672	24.12847	24.31182	28.52202	28.32875	29.85547
95	10.61207	10.30952	9.54470	21.35730	21.18952	21.35874	25.79067	25.59277	27.15286
100	7.28423	7.20730	6.34529	18.09945	18.46754	18.65030	22.84229	23.00171	24.63365
	$rho = 0$, $v0 = 0.05$			$\rho = -0.5$, $v0 = 0.1$			$\rho = -0.5$, $v0 = 0.1$		

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	**0.5**			**2**			**3**		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Full	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**
90	14.06407	13.91363	13.50851	24.41388	24.05615	24.35216	29.00286	29.18680	29.88640
95	10.24581	10.25548	9.66329	21.79879	20.96414	21.40435	26.28235	26.37107	27.18698
100	7.39412	7.21322	6.50357	19.11952	18.82747	18.70001	23.64200	23.60276	24.67067
	$\rho = 0$, $v_0 = 0.05$			$\rho = -0.5$, $v_0 = 0.1$			$\rho = -0.5$, $v_0 = 0.1$		

Table 9: Call prices under regime-changing and rough volatility

(a) Starting in a low state: $\theta_0 = \theta^1$

K/T	1			2			5		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Full	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**
90	17.42360	16.98568	17.12488	23.39505	23.50787	23.43841	36.50005	36.75730	39.42251
95	13.78313	13.97298	13.66086	19.78577	20.39672	20.37560	33.44811	34.37670	37.10921
100	10.88857	11.07770	10.58581	17.24397	17.47267	17.57232	31.72467	31.98527	34.91732
	$\rho = -0.5$, $v_0 = 0.05$			$\rho = -0.5$, $v_0 = 0.05$			$\rho = -0.5$, $v_0 = 0.1$		

(b) Starting in a high state: $\theta_0 = \theta^2$

K/T	1			2			5		
	Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
	Full	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**	**Full**	**Partial**	**Fourier**
90	18.07010	18.41721	17.18695	23.39505	23.02899	23.48063	37.63668	36.75730	39.44308
95	13.89200	14.44758	13.73703	19.78577	19.78565	20.42420	34.18855	34.37670	37.13149
100	10.89707	11.24528	10.67650	17.24397	17.83789	17.62695	32.68452	31.98527	34.94115
	$\rho = -0.5$, $v_0 = 0.05$			$\rho = -0.5$, $v_0 = 0.05$			$\rho = -0.5$, $v_0 = 0.1$		

Table 10: Call prices under Heston model modifications, maturity fixed at $T = 1$

(a) Starting in a low state: $\theta_0 = \theta^1$

Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
24.18566	0.00023	24.40145	24.87668	0.12432	24.41194	24.95195	0.14809	24.54135
19.73528	0.00033	20.01877	20.64484	0.11906	20.03026	20.87358	0.14293	20.28785
15.54209	0.00042	15.86069	15.91675	0.10802	15.87262	16.74382	0.13051	16.32680
11.74930	0.00050	12.02694	12.01290	0.10022	12.03865	12.74631	0.11821	12.64163
8.50614	0.00055	8.63622	8.28652	0.08782	8.64671	9.40263	0.11208	9.30234
5.91152	0.00058	5.80791	5.03983	0.07613	5.81549	6.43873	0.09919	6.46152
3.97837	0.00058	3.62625	2.88119	0.06492	3.62874	3.86032	0.08174	4.24546
Classical Heston			**Rough Heston**			**Regime Switching Rough Heston**		

(b) Starting in a high state: $\theta_0 = \theta^2$

Monte Carlo		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**	**Monte Carlo**		**Semi-Analytic**
Price	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**	**Price**	**std Error**	**Fourier**
25.24961	0.00025	25.56759	26.12745	0.18946	25.64066	25.74824	0.05678	24.97705
21.34310	0.00028	21.66156	22.12496	0.18115	21.74460	21.70714	0.05376	20.86047
17.77426	0.00030	18.04638	17.86913	0.16392	18.13614	17.96501	0.05086	17.03892
14.58986	0.00032	14.76582	14.37190	0.15299	14.85821	14.15027	0.04681	13.55432
11.81316	0.00034	11.85344	11.37817	0.13983	11.94405	11.00267	0.04286	10.45512
9.44701	0.00035	9.32861	8.53524	0.12659	9.41330	8.10423	0.03822	7.79629
7.47475	0.00035	7.19422	6.24291	0.11296	7.26965	5.76181	0.03393	5.61828
Classical Heston			**Rough Heston**			**Regime Switching Rough Heston**		

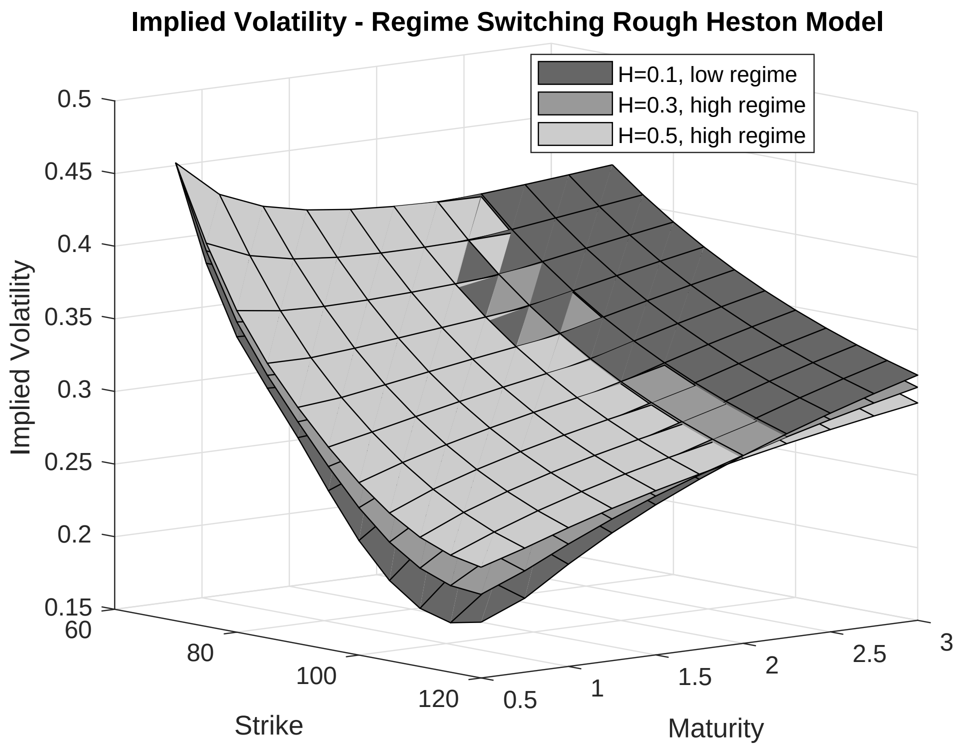


Figure 2: Implied volatility surface for changing Hurst parameter H under different initial volatility regime

Finally, we proceed to show the computational speed across the modeling frameworks and numerical methods adopted. All codes were written in MATLAB R2017b and run on a Dell *Intel(R) Core(TM) i5 - 3.30GHz and with 16 GB memory*. Table 11 summarizes the results.

Table 11: Computational speed comparison, 10 option prices, time in seconds.

	Pricing Method		
Model	Semi-Analytic	Full Monte Carlo	Partial Monte Carlo
Classical Heston	0.534319	8.385977	-
Regime Switching Heston	5.065121	30.461079	-
Rough Heston	4.041738	837.789867	-
Regime Switching Rough Heston	19.56441	823.823312	14.701957

4 Conclusion

In this paper we studied the regime switching rough Heston models. It combines the recently developed theory of rough volatility using rough Riccati-equations as in Euch and Rosenbaum (2016) and the regime switching extension of the Heston model as in Elliott et al. (2016). The main goal is to develop a tractable model that accounts for the two important stylized features of volatility simultaneously, namely the rough behaviour in its local behaviour, and the regime switching property consistent with more long term economic consideration.

Since there isn't yet any model combining rough Brownian motion with jumps and because of the analytic tractability we opted for the regime switching using hidden Markov chain instead of jumps. This enables us to incorporate sudden changes even in the rough volatility case.

The call option price is still given in a semi-analytic formula. We developed a pricing engine and fully implemented this analytic approach to the rough switching Heston model.Two simulation based methods has been developed. The first is the full Monte-Carlo-Simulation of the underlying stochastic process and the second one is just the simulation of the regime switching Markov process, then applying the Riccati equation and the Fourier methods for the call option. The latter is shown to be even faster than the semi-analytic formula as it avoids having to compute the resolvent matrix with is time consuming.

Our results show that the deviation between the approaches are small and consistent for any given time to expiry. We also analyse sensitivity to several input parameters. In particular, we show the sensitivity with respect to roughness, the Hurst parameter H. Actually this sensitivity depends on the time to expiry of the option. Concerning the regime switches we analyse Q-matrices, one with only one change per year as carried out in Elliott et al. (2016) and one with fast changes, approximately 5 per year. There is only a slight impact on call prices, see also Tables 5, 7 and Table 10.

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