

# Introduction to Variance Swaps

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## Abstract

The purpose of this article is to introduce the properties of variance swaps, and give insights into the hedging and valuation of these instruments from the particular lens of an option trader.

- Section 1 gives general details about variance swaps and their applications.
- Section 2 explains in 'intuitive' financial mathematics terms how variance swaps are hedged and priced.

## Keywords

Variance swap, volatility, path-dependent, gamma risk, static hedge.

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## Payoff

A variance swap is a derivative contract which allows investors to trade future realized (or historical) volatility against current implied volatility. The reason why the contract is based on variance—the squared volatility—is that only the former can be replicated with a static hedge, as explained in the penultimate Section of this article.

Sample terms are given in Exhibit 1 in the next page.

These sample terms reflect current market practices. In particular:

1. Asset returns are computed on a logarithmic basis rather than arithmetic.
2. The mean return, which appears in the habitual statistics formula for variance, is ditched. This has the benefit of making the payoff perfectly additive (i.e. 1-year variance can be split into two 6-month segments.)
3. The 252 scaling factor corresponds to the standard number of trading days in a year. The  $10,000 = 100^2$  scaling factor corresponds to the conversion from decimal (0.01) to percentage point (1%).

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4. The notional is specified in volatility terms (here €50,000 per 'vega' or volatility point.) The true notional of the trade, called variance notional or variance units, is given as:

$$\text{Variance Notional} = \frac{\text{Vega Notional}}{2 \times \text{Strike}}$$

With this convention, if realized volatility is 1 point above the strike at maturity, the payoff will approximately be equal to the Vega Notional.

## Variance Swaps vs. Volatility Swaps

The fair strike of a variance swap is slightly higher than that of a volatility swap. This is to compensate for the fact that variance is convex in volatility, as illustrated in Exhibit 2 in the next page. Identical strikes for the two instruments would otherwise lead to an arbitrage.

Intuitively, the difference in fair strikes is related to the volatility of volatility: the higher the 'vol of vol', the more expensive the convexity effect of variance<sup>1</sup>. This phenomenon is clearly observed when the implied volatility skew is steep, as skew accounts for the empirical fact that volatility is

Exhibit 1—Variance Swap on Dow Jones Euro Stoxx 50 Index: sample terms and conditions

General Terms	
Swap Buyer (Party A)	TBD [e.g. Investor]
Swap Seller (Party B)	TBD [e.g. Dresdner Bank AG]
Trade date	TBD
Expiration Date	TBD [e.g. Trade date + 1 year]
Swap	Variance Swap on Equity Index
Index	.STOXX50E
Currency	EUR
Observation Frequency	Daily
Approximate Vega Notional	EUR 50,000
Variance Units	1157.41 (Approximate Vega Notional divided by 2 x Volatility Strike)
Volatility Strike	21.6
Variance Strike	466.56 (square of the Volatility Strike)
Cash Settlement Date	Two business days after the Expiration Date
Payment Amount	The Payment Amount is calculated as:  $\text{Variance Units} \times (\sigma^2 - \text{Variance Strike})$ If the Payment Amount is positive, the Swap Seller (B) pays to the Swap Buyer (A); if the Payment Amount is negative, the Swap Buyer (A) pays to the Swap Seller (B) the absolute value.
Variance Calculation	$\sigma^2$ is the observed realized variance of the Index between the Trade Date and the Expiration Date, given as:  $\sigma^2 = \frac{10,000 \times 252 \times \sum_{i=1}^{N_{\text{Actual}}} \text{Return}_i^2}{N_{\text{Expected}}}$ Where:  $\text{Return}_i = \ln \left( \frac{E_i}{E_{i-1}} \right)$ $N_{\text{Expected}}$ is the expected number of trading days from, but not including, the Trade Date, up to and including the Expiration Date. $N_{\text{Actual}}$ is the actual number of trading days on which no market disruption event occurs from, but not including, the Trade Date, up to and including the Expiration Date. $E_0$ is the closing level of the index on Trade Date. $E_i$ is the closing level of the index on date $i$ or, at Expiration Date, the options final settlement level.

non constant. In fact, the fair strike of variance is often in line with the implied volatility of the 30% delta put.

### Rule of Thumb

Demeterfi–Derman–Kamal–Zou (1999) derive the following rule of thumb when skew is linear in strike:

$$K_{\text{var}} \approx \sigma_{\text{ATMF}} \sqrt{1 + 3T \times \text{skew}^2}$$

where  $\sigma_{\text{ATMF}}$  is the at-the-money-forward volatility,  $T$  is the maturity, and  $\text{skew}$  is the slope of the skew curve. For example, with  $\sigma_{\text{ATMF}} = 20\%$ ,  $T = 2$  years, and a 90–100 skew of 2 vegas, we have  $K_{\text{var}} \approx 22.3\%$ . In comparison, a 30% delta put would have an implied volatility of 22.2% assuming a linear skew.

However, this rule of thumb becomes inaccurate when skew is steep.

## Applications

### Bets on Future Realized Volatility

Variance swaps are ideal instruments to bet on volatility:

- Unlike vanilla options, variance swaps do not require any delta-hedging
- Unlike the P&L of a delta-hedged vanilla option, the payoff at maturity of a long variance position will always be positive when realized volatility exceeds the strike<sup>2</sup>. (See the next Section on the path-dependency of vanilla options for more details.)
- The sensitivity of a variance swap to changes in (squared) implied volatility linearly collapses through time.

Furthermore, volatility sellers will find variance swaps more attractive than at-the-money options due to their higher variance strike. However this excess profit reflects the higher risk in case realized volatility jumps well above the strike.

### Bets on Forward Realized Volatility

Forward-starting variance swaps can be synthesized with a calendar spread of two spot-starting variance swaps, with appropriate notionals. This is because the variance formula is designed to be perfectly additive. Taking annualization into account, we can indeed write:

$$3 \times \text{Realized}_{3Y} = \text{Realized}_{1Y} + 2 \times \text{Forward Realized}_{1Y \times 2Y}$$

where  $\text{Realized}_{1Y}$  is the future 1-year realized volatility,  $\text{Realized}_{3Y}$  is the future 3-year realized volatility, and  $\text{Forward Realized}_{1Y \times 2Y}$  is the future 2-year realized volatility starting in 1 year.

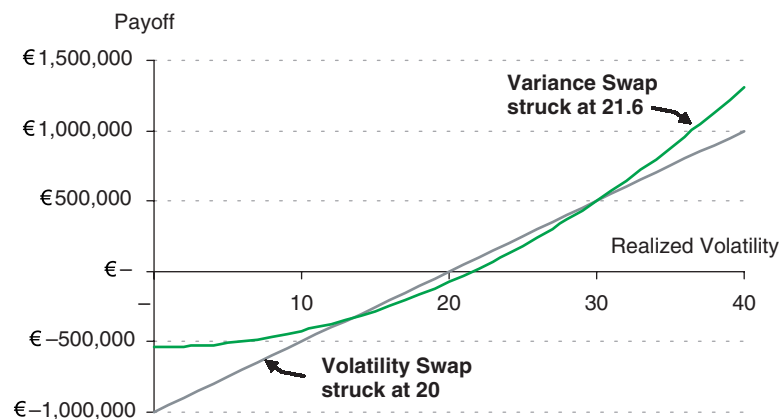
Thus, for a given forward variance notional, we must adjust the spot variance notionals as follows:

$$\text{Variance Notional}_{1Y} = \frac{1 \times \text{Forward Variance Notional}_{1Y \times 2Y}}{2}$$

$$\text{Variance Notional}_{3Y} = \frac{3 \times \text{Forward Variance Notional}_{1Y \times 2Y}}{2}$$



**Exhibit 2—Variance swaps are convex in volatility**



The resulting implicit fair strike for the forward variance swap is:

$$\sqrt{\frac{3 \times K_{3Y \text{ var}}^2 - 1 \times K_{1Y \text{ var}}^2}{2}}$$

For example, with  $K_{1Y \text{ var}} = 18.5$ ,  $K_{3Y \text{ var}} = 19.5$ , the fair strike of a 2-year variance swap starting in 1 year would be:

$$\sqrt{\frac{3 \times 19.5^2 - 1 \times 18.5^2}{2}} \approx 20.0$$

The corresponding replication strategy for a long €100,000 forward vega notional position (equivalent to 2,500 forward variance units) would be to buy  $3 \times 2,500/2 = 3,750$  variance units of the 3-year variance swap and sell  $2,500/2 = 1,250$  variance units of the 1-year.

## Correlation Trading

By simultaneously selling a variance swap on an index and buying variance swaps on the constituents, an investor effectively takes a short position on realized correlation. This type of trade is known as a variance dispersion. A proxy for the implied correlation level sold through a variance dispersion trade is given as the squared ratio of the index variance strike to the average of constituents' variance strikes.

Note that in order to offset the vega exposure between the two legs we must adjust the vega notionals of the constituents by a factor equal to the square root of implied correlation. It can be shown that by dynamically trading vega-neutral variance dispersions until maturity we would almost replicate the payoff of a correlation swap<sup>3</sup>.

## Trading P&L of a Vanilla Option

### Daily Trading P&L

Following the theory developed by Black–Scholes and Merton in 1973, the sensitivity of a vanilla option to changes in the stock price, or delta, can be completely offset by holding a reverse position in the stock in quantity equal to the delta. The iteration of this strategy until maturity is known as delta-hedging.

Once the delta is hedged, an option trader is primarily left with three risks:

- Gamma: sensitivity of the option delta to changes in the underlying stock price;
- Theta or time decay: sensitivity of the option price to the passage of time;
- Vega: sensitivity of the option price to changes in the market's expectation of future volatility (i.e. implied volatility.)

We can break down the daily P&L on a delta-neutral option position along these risks:

$$\text{Daily P\&L} = \text{Gamma P\&L} + \text{Theta P\&L} + \text{Vega P\&L} + \text{Other} \quad (\text{Eq. 1})$$

Here 'Other' includes the P&L from financing the reverse delta position on the underlying, as well as the P&L due to changes in interest rates, dividend expectations, and high-order sensitivities (e.g. sensitivity of Vega to changes in stock price, etc.)

Using Greek letters, we can rewrite Equation 1 as:

$$\text{Daily P\&L} = \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) + V \times (\Delta \sigma) + \dots$$

where  $\Delta S$  is the change in the underlying stock price,  $\Delta t$  is the fraction of time elapsed (typically 1/365), and  $\Delta \sigma$  is the change in implied volatility.

Assuming a zero interest rate, constant volatility and negligible high-order sensitivities, we can further reduce this equation to the first two terms:

$$\text{Daily P\&L} = \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) \quad (\text{Eq. 2})$$

Equation 2 can be further expanded to be interpreted in terms of realized and implied volatility. This is because in our zero-interest rate world Theta can be re-expressed with Gamma<sup>4</sup>:

$$\Theta = -\frac{1}{2}\Gamma S^2 \sigma^2 \quad (\text{Eq. 3})$$

Plugging Equation 3 into Equation 2, we obtain a characterization of the daily P&L in terms of squared return and squared implied volatility:

$$\text{Daily P\&L} = \frac{1}{2}\Gamma S^2 \times \left[ \left( \frac{\Delta S}{S} \right)^2 - \sigma^2 \Delta t \right] \quad (\text{Eq. 4})$$

The first term in the bracket,  $\frac{\Delta S}{S}$ , is the percent change in the stock price—in other words, the one-day stock return. Squared, it can be interpreted as the *realized* one-day variance. The second term in the bracket,  $\sigma^2 \Delta t$ , is the squared daily *implied* volatility, which one could name the daily implied variance. Finally, the factor in front of the bracket,  $\frac{1}{2}\Gamma S^2$ , is known as *Dollar Gamma*: an adjusted measure for the second-order sensitivity of the option price to a squared percent change in the stock price.

In short, Equation 4 tells us that the daily P&L of a delta-hedged option position is driven by the difference between realized and implied variance, multiplied by the Dollar Gamma.

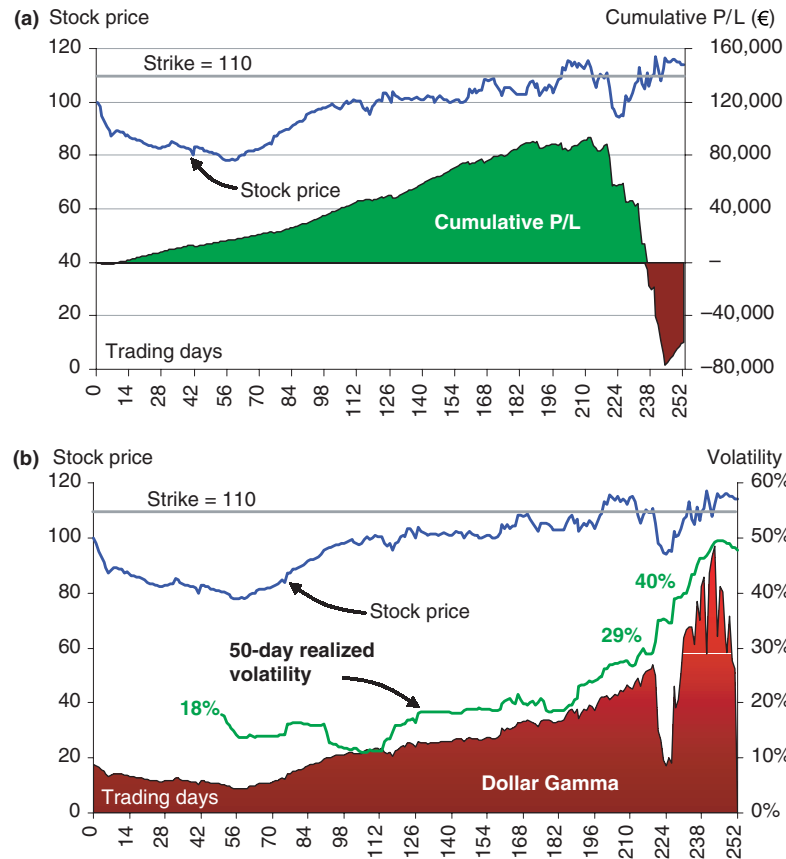
### Path Dependency

One can already see the connection between Equation 4 and variance swaps: if we sum all daily P&L's until maturity, we have an expression for the final trading P&L on a delta-neutral option position:

$$\text{Final P\&L} = \sum_{t=0}^n \gamma_t [r_t^2 - \sigma^2 \Delta t] \quad (\text{Eq. 5})$$

**Exhibit 3—Path-dependency of the cumulative P&L for a dynamically hedged option position**

In this simulation a trader issued 25,000 1-year calls struck at €110 for an implied volatility of 30%, and followed a daily delta-hedging strategy. The 1-year realized volatility at maturity was 27.6%, yet the cumulative trading P&L was down €60k. In figure (a) we can see that the strategy was up €100,000 two months before maturity and suddenly dropped. In figure (b) we can see that in the final two months the 50-day realized volatility rose well above 30% while the (short) dollar gamma peaked. Because the daily P&L of an option position is weighted by the dollar gamma, and because the volatility spread between implied and realized was negative, the final P&L plunged, even though the 1-year realized volatility was below 30%!



where the subscript  $t$  denotes time dependence,  $r_t$  is the stock daily return at time  $t$ ,  $\gamma_t$  is the dollar gamma, and  $n$  is the number of trading days until maturity.

Equation 5 is close to the payoff of a variance swap: it is a weighted sum of squared realized returns minus a constant that has the same role as a strike. But in a variance swap the weights are constant, while here the weights depend on the option gamma through time. This explains an option trading phenomenon known as path-dependency, illustrated in Exhibit 3.

**Static Replication of Variance Swaps**

In the previous section, we saw that a trader who follows a delta-hedging strategy is basically replicating the payoff of a weighted variance swap,

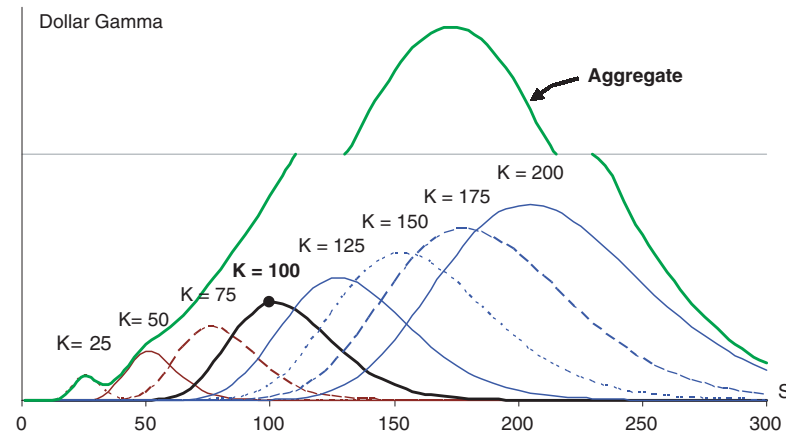
with weights equal to the dollar gamma. This result also holds for a portfolio of options. If we could find a combination of calls and puts such that their aggregate dollar gamma is always constant, we would have a semi-static hedge for variance swaps<sup>5</sup>.

Exhibit 4 shows the dollar gamma of options with various strikes in function of the underlying level. We can see that the contribution of low-strike options to the aggregate gamma is small compared to high-strike options. Hence, we need to increase the weights of low-strike options and decrease the weights of high-strike options.

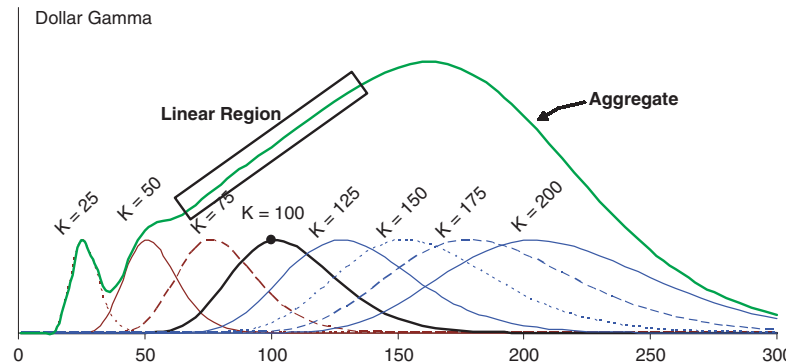
One ‘naïve’ idea is to use weights inversely proportional to the strike so as to scale all individual dollar gammas to the same peak level<sup>6</sup>, as illustrated in Exhibit 5. We can see that the aggregate dollar gamma is still non-constant, but we can notice the existence of a linear region.

This observation is crucial: if we can regionally obtain a linear aggregate dollar gamma with a certain weighting scheme  $w(K)$ , then the transformed weights  $w'(K) = w(K)/K$  will produce a constant dollar gamma in that region. Since the naïve weights are inversely proportional to the strike  $K$ , the correct weights should be chosen to be inversely proportional to the squared strike, i.e.:  $w'(K) = 1/K^2$ .

**Exhibit 4—Dollar Gamma of vanilla options with strikes 25 to 200 spaced 25 apart**



**Exhibit 5—Weighted Dollar Gamma of vanilla options: weights inversely proportional to the strike K**



**Exhibit 6—Weighted Dollar Gamma of vanillas: weights inversely proportional to the square of strike**

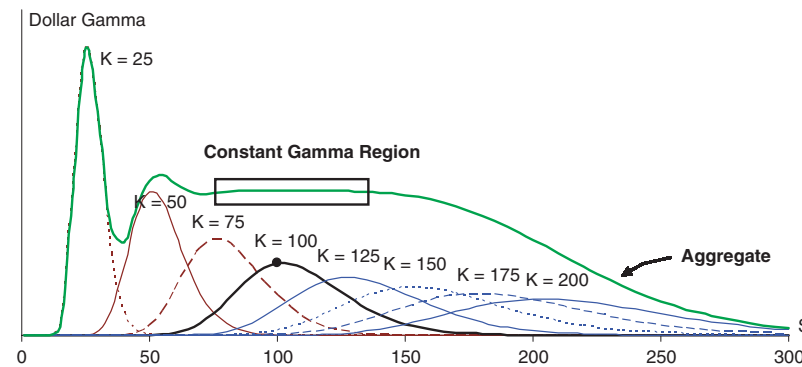


Exhibit 6 shows the results for the transformed weights: as expected, we now have a constant region for the aggregate dollar gamma.

To obtain a perfect constant aggregate gamma through all underlying levels would take infinitely many options struck along a continuum between 0 and infinity.

### Interpretation

Having mentioned that a variance swap could be perfectly hedged with an infinite portfolio of puts and calls of constant Dollar Gamma, one might want to view this portfolio as a new kind of derivative and speculate on its nature. Denoting  $f$  the price of the derivative and  $S$  that of the underlying, the Dollar Gamma is given as:

$$\Gamma^S(S) = \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \times S^2$$

It follows that a constant Dollar Gamma derivative satisfies:

$$\frac{\partial^2 f}{\partial S^2} = \frac{a}{S^2}$$

for some constant  $a$ . The solution to this second-order differential equation is of the form:

$$f(S) = -a \ln(S) + bS + c$$

where  $a, b, c$  are constants and  $\ln(\cdot)$  is the natural logarithm.

This means that the perfect hedge for a variance swap would be a contract paying the log-price of the underlying stock at maturity, and a combination of the stock and cash. Unfortunately such log-contract does not trade—or, rather, it trades in the format of a variance swap.

## Valuation

In the absence of arbitrage, the fair market value of a variance swap must equal the price of its replicating portfolio of puts and calls. As such, no model specification is required: the theoretical price can be calculated for any reasonable volatility smile. Note, however, that this approach does not take into account the impact of jumps or discrete dividend payments.

Given a set of  $N$  strikes  $(k_1, k_2, \dots, k_n, \dots, k_N)$  where  $k_n = 1$  denotes the split between out-of-the-money-forward puts and calls, a quick proxy for

**Exhibit 7—Fair value decomposition of a variance swap through a replicating portfolio of puts and calls**

In this example, we consider a variance swap on the S&P 500 index expiring on 15 December 2006. The time to maturity  $T$  is 1.1032 and the discount factor to maturity is  $DF = 0.94889$ . The total cost of the replicating portfolio (i.e. the weighted sum of put and call prices multiplied by  $2/T$ ) is 2.45%, which corresponds to a fair strike of 16.06%. A more accurate model gave 15.83%.

Weight = 5% Strike% <sup>2</sup>	Call / Put	Strike	Strike (%Forward)	Implied Vol	Price (%Forward)
20.00%	Put	629.88	50.0%	24.53	0.02%
16.53%	Put	692.87	55.0%	24.39	0.06%
13.89%	Put	755.86	60.0%	23.91	0.14%
11.83%	Put	818.85	65.0%	23.06	0.28%
10.20%	Put	881.83	70.0%	21.99	0.48%
8.89%	Put	944.82	75.0%	20.81	0.78%
7.81%	Put	1,007.81	80.0%	19.56	1.22%
6.92%	Put	1,070.80	85.0%	18.26	1.84%
6.17%	Put	1,133.79	90.0%	16.92	2.72%
5.54%	Put	1,196.78	95.0%	15.55	3.94%
2.50%	Put	1,259.76	100.0%	14.20	5.64%
2.50%	Call	1,259.76	100.0%	14.20	5.64%
4.54%	Call	1,322.75	105.0%	13.12	3.30%
4.13%	Call	1,385.74	110.0%	12.34	1.73%
3.78%	Call	1,448.73	115.0%	11.83	0.82%
3.47%	Call	1,511.72	120.0%	11.55	0.37%
3.20%	Call	1,574.71	125.0%	11.45	0.16%
2.96%	Call	1,637.69	130.0%	11.40	0.06%
2.74%	Call	1,700.68	135.0%	11.38	0.03%
2.55%	Call	1,763.67	140.0%	11.37	0.01%
2.38%	Call	1,826.66	145.0%	11.38	0.00%
2.22%	Call	1,889.65	150.0%	11.39	0.00%

Source: DrKW.

the fair value of a variance swap with \$1 variance notional is:

$$\begin{aligned} \text{VarSwap}_{FV} \approx & \frac{2}{T} \left[ \sum_{i=1}^n \frac{(k_i - k_{i-1})}{k_i^2} \text{put}_{\%F}(k_i) \right. \\ & \left. + \sum_{i=n+1}^N \frac{(k_i - k_{i-1})}{k_i^2} \text{call}_{\%F}(k_i) \right] - DF(0, T) \times K_{\text{var}}^2 \end{aligned}$$

where  $T$  is the maturity,  $K_{\text{var}}$  is the variance strike,  $DF(0, T)$  is the present value of \$1 collected at maturity,  $\text{put}_{\%F}(k)$  or  $\text{call}_{\%F}(k)$  is the price of a European put or call struck at  $k$ , and  $k_0 = 0$ . Note that the strikes and option prices are expressed in percentage of the forward price.

Exhibit 7 above gives a calculation example with strikes between 50% and 150% spaced 5% apart.

## Conclusion

### Looking Forward

Variance swaps have become an increasingly popular type of 'light exotic' derivative instrument. Market participants are the major derivatives houses, hedge funds, and institutional investors. An unofficial estimate of the typical inter-broker trading volume is between \$1,000,000 and \$7,000,000 total vega notional in the European and American markets every day.

With the commoditization of variance swaps, variance is becoming an asset class of its own. A number of volatility indices have been launched or

adjusted to follow the weighting methodology of the replicating portfolio, in particular the new Chicago Board Options Exchange SPX Volatility Index (VIX) and the Deutsche Börse VSTOXX Volatility Index. The current hot development is options on realized volatility, with recent research results by Dunamu (2004) and Carr-Lee (2005).

## ACKNOWLEDGEMENTS

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## FOOTNOTES & REFERENCES

1. Readers with a mathematical background will also recall Jensen's inequality:  $E(\sqrt{\text{variance}}) \leq \sqrt{E(\text{variance})}$ .
2. This is because the sign of  $(R^2 - K^2) = (R - K)(R + K)$  is determined by  $R - K$ , where  $R$  is the realized volatility and  $K$  is the strike.
3. A correlation swap is a derivative contract on several assets where counterparties exchange a fixed cash flow against a variable amount equal to the notional multiplied by the average of the pair-wise correlation coefficients between the assets.

4. In a zero interest-rate world the Black-Scholes partial differential equation becomes:  $\frac{1}{2}S^2\sigma^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} = 0$ .

5. The hedge is semi-static because the portfolio of puts and calls still needs to be delta-hedged. However, no dynamic trading of options is required.

6. This is because the dollar gamma peaks around the strike. Specifically, it can be shown that the peak is reached when the stock price is equal to  $S^* = Ke^{\sigma\sqrt{T} - \sigma^2 T/2} \approx K$ , with a peak level proportional to  $S^*$ .

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