Model-Free Pricing and Hedging of Forward Starting Volatility Swaps

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Abstract

In this paper we give a model-free approximation for the price of forward starting volatility swaps. Moreover, we show that a self-financing and model-independent approximate hedge is achieved by dynamically trading zero vanna forward starting straddles with a skew adjusted notional. To the best of our knowledge our result is the first non-parametric result for pricing and hedging of forward starting volatility swaps.

1 Preliminaries

In Rolloos and Arslan [2017] it is proved that for a wide class of stochastic volatility models the fair strike of a spot starting volatility swap is to a good approximation equal to the implied volatility of a vanilla option corresponding to the strike where the Black-Merton-Scholes (BMS) vanna and volga of the option is zero. This note generalises the result to forward starting volatility swaps. Furthermore, we show that a self-financing model-independent hedge for forward starting volatility swaps can be achieved by dynamically rebalancing zero vanna straddles with a skew dependent notional. To the best of our knowledge, our result is the first non-parametric result on pricing and hedging of forward starting volatility swaps.

In the following it is assumed that the underlying index follows a stochastic volatility process of the form

$$\begin{aligned} dS_t &= \sigma_t S_t \left(\rho dW_t^{\sigma} + \bar{\rho} dW_t^{\perp} \right) \\ d\sigma_t &= a(\sigma_t, t)dt + b(\sigma_t, t)dW_t^{\sigma} \end{aligned} \tag{1.1}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, $dW^{\sigma}dW^{\perp} = 0$, and $a(\sigma_t, t)$ and $b(\sigma_t, t)$ are deterministic functions of time and volatility. We have set interest rates and dividend yields to zero, but the gist of the story and results will also hold for nonzero deterministic rates and dividends.

2 Forward starting options

In addition to a liquid market for vanilla options we will also assume that forward starting European call and put options are traded in sufficient volumes. For major equity indices and FX pairs this is a reasonable premise. There are two common flavours of forward starting options. Let T denote the forward start date (also known as the fixing date), T' the expiry date, and k the forward start strike, then the payoff of the the first type of forward starting option is

$$(S_{T'} - kS_T)_+ \tag{2.1}$$

while the second type has payoff

$$\frac{(S_{T'} - kS_T)_+}{S_T} = \left(\frac{S_{T'}}{S_T} - k \right)_+ \tag{2.2}$$

Forward starting options with payoff given by (2.1) are also known as *asset* forward starting options, and those with payoff given by (2.2) are called *cash* forward starting options. In this note we will consider cash forward starting options, and future reference to forward starting options will mean options with payoff given by (2.2), unless explicitly stated otherwise.

The price today of a cash forward starting option is

$$C^{FS}(t, k, T, T') = E_t \left[\frac{(S_{T'} - kS_T)_+}{S_T} \right] \tag{2.3}$$

Using the tower law we can write

$$E_t\left[\frac{(S_{T'}-kS_T)_+}{S_T}\right]=E_t\left[E_T\left[\frac{(S_{T'}-kS_T)_+}{S_T}\right]\right]=E_t\left[\frac{C(S_T,T,kS_T,T')}{S_T}\right] \tag{2.4}$$

where $C(S_T, T, kS_T, T')$ is the price of a plain vanilla option at time T with expiry date T'. Hence,

$$C^{FS}(t,k,T,T')=E_t\left[\frac{C(S_T,T,kS_T,T')}{S_T}\right] \tag{2.5}$$

For $t \geq T$ the price of the forward starting option is equal to the price of a vanilla option as S_T will then be known and therefore the strike will be fixed. In the following we will concentrate on the price of the forward starting option for $t < T$.

Equation (2.5) holds for general stochastic volatility models (1.1). A specific case of (1.1) is the BMS model. Under the assumptions of the BMS model the price of a forward starting European call option is

$$C^{FS,BMS}(t,k,T,T')=E_t\left[\frac{C^{BMS}(S_T,T,kS_T,T')}{S_T}\right] \tag{2.6}$$

where $C^{BMS}(S_T, T, kS_T, T')$ is the BMS price of a plain vanilla call at time T with maturity date T'. Since the BMS formula is homogenous of degree one, and volatility is constant in a BMS world, (2.6) can be simplified to

$$C^{FS,BMS}(t,k,T,T')=C^{BMS}(1,T,k,T') \tag{2.7}$$

The BMS vanilla call option price $C^{BMS}(1, T, k, T')$ should be interpreted as a call option price with spot always at 1, strike k, and time to maturity always equal to $T' - T$. So in a BMS world the price of a cash forward start option is constant $\forall t < T$.

Even though the assumptions underlying the BMS model are violated we can always express the stochastic volatility price of a forward starting option in terms of the forward start BMS price by an appropriate choice of implied volatility:

$$C^{FS}(t,k,T,T')=C^{FS,BMS}(t,k,T,T',\Sigma^{FS})=C^{BMS}(1,T,k,T',\Sigma^{FS}) \tag{2.8}$$

The implied volatility is a function of strike k, running time t, forward start date T, expiry date T', as well as other variables and parameters such as the instantaneous volatility σ_t and correlation ρ. As the cash forward start option does not depend on S_t, Σ^{FS} is also independent of the spot price.

3 The generalised Hull-White formula

To derive the generalised Hull-White formula for forward starting options, we first apply the formula to $C(S_T, T, kS_T, T')$ in equation (2.5) at T:

$$C(S_T,T,kS_T,T')=E_T\left[C^{BMS}(S_T M_{T,T'}(\rho),T,kS_T,T',\bar{\rho}\sigma_{T,T'})\right] \tag{3.1}$$

with

$$M_{T,T'}(\rho) = \exp\left\{-\frac{\rho^2}{2}\int_T^{T'}\sigma_u^2 du + \rho\int_T^{T'}\sigma_u dW_u^\sigma\right\} \tag{3.2}$$

and $\sigma_{T,T'}$ is the future realized volatility over $[T, T']$:

$$\sigma_{T,T'} = \sqrt{\frac{1}{T'-T}\int_T^{T'}\sigma_u^2 du} \tag{3.3}$$

We can Taylor expand the expression in brackets in (3.1) around $\rho = 0$:

$$\begin{aligned} C^{BMS}(S_T M_{T,T'}(\rho), T, kS_T, T', \bar{\rho}\sigma_{T,T'}) &\approx S_T C^{BMS}(1, T, k, T', \sigma_{T,T'}) \\ &\quad + \rho S_T \Delta^{BMS}(1, T, k, T', \sigma_{T,T'})\int_T^{T'}\sigma_u dW_u^\sigma \end{aligned} \tag{3.4}$$

where we have made use of the homogeneity of the BMS formula. Hence,

$$\begin{aligned} \frac{C(S_T, T, kS_T, T')}{S_T} &\approx E_T\left[C^{BMS}(1, T, k, T', \sigma_{T,T'})\right] \\ &\quad + \rho E_T\left[\Delta^{BMS}(1, T, k, T', \sigma_{T,T'})\int_T^{T'}\sigma_u dW_u^\sigma\right] \end{aligned} \tag{3.5}$$

Taking now the expectation at current time t of the above equation, The price of a forward starting option (2.5) is then

$$\begin{aligned} C^{FS}(t, k, T, T') &\approx E_t\left[C^{BMS}(1, T, k, T', \sigma_{T,T'})\right] \\ &\quad + \rho E_t\left[\Delta^{BMS}(1, T, k, T', \sigma_{T,T'})\int_T^{T'}\sigma_u dW_u^\sigma\right] \end{aligned} \tag{3.6}$$

Using equation (2.8) we can equivalently write

$$\begin{aligned} C^{BMS}(1, T, k, T', \Sigma_k^{FS}) &\approx E_t\left[C^{BMS}(1, T, k, T', \sigma_{T,T'})\right] \\ &\quad + \rho E_t\left[\Delta^{BMS}(1, T, k, T', \sigma_{T,T'})\int_T^{T'}\sigma_u dW_u^\sigma\right] \end{aligned} \tag{3.7}$$

4 Forward starting volatility swaps

4.1 Pricing

According to equation (2.8) the price of a forward start option can always be expressed in terms of a BMS vanilla price with an appropriate choice of implied volatility. We will choose a forward start option with a specific strike, namely one with a strike k_{d_2} and implied volatility $\Sigma_{d_2}^{FS}$ such that

$$d_2 = \frac{\ln 1/k_{d_2}}{\Sigma_{d_2}^{FS}\sqrt{T'-T}} - \frac{\Sigma_{d_2}^{FS}\sqrt{T'-T}}{2} = 0 \tag{4.1}$$

When $d_2 = 0$ the BMS vanna and volga of the option are zero. In the following we will use the notation ν^{BMS}, va^{BMS}, and vo^{BMS} for the BMS vega, vanna, and volga of the option.

Following the method in Rolloos and Arslan [2017], we Taylor expand the two terms on the right hand side of the approximation in (3.7) around the implied volatility $\Sigma_{d_2}^{FS}$. This gives us

$$\begin{aligned} C^{BMS}(1,T,k_{d_2},T',\sigma_{T,T'}) &\approx C^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) \\ &\quad + \nu^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS})(\sigma_{T,T'} - \Sigma_{d_2}^{FS}) \\ &\quad + \frac{1}{2} vo^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS})(\sigma_{T,T'} - \Sigma_{d_2}^{FS})^2 \\ &= C^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) \\ &\quad + \nu^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS})(\sigma_{T,T'} - \Sigma_{d_2}^{FS}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\Delta^{BMS}(1,T,k_{d_2},T',\sigma_{T,T'}) \\ &\qquad \approx \Delta^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) \\ &\qquad\quad + va^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS})(\sigma_{T,T'} - \Sigma_{d_2}^{FS}) \\ &\qquad = \Delta^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) \end{aligned} \tag{4.3}$$

where we have made use of the fact

$$va^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) = vo^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) = 0 \tag{4.4}$$

Equation 3.7 now simplifies to

$$\begin{aligned} C^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) &\approx C^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) \\ &\quad + \nu^{BMS}(1,T,k_{d_2},T',\Sigma_{d_2}^{FS}) E_t\left[\sigma_{T,T'} - \Sigma_{d_2}^{FS}\right] \end{aligned} \tag{4.5}$$

This can only be the case if

$$\boxed{E_t\left[\sigma_{T,T'}\right] \approx \Sigma_{d_2}^{FS}} \tag{4.6}$$

This equation is the generalisation to forward starting volatility swaps of equation (40) in Rolloos and Arslan [2017].

4.2 Hedging

From equation (4.6) it is clear that the hedge for forward starting volatility swaps is approximately the change in the zero vanna implied volatility. The main problem that needs to be solved is that an implied volatility which is initially zero vanna at time t will not be a zero vanna implied volatility at time $t + dt$, when the strike is fixed. If the initial zero vanna strike is k_{d_2}, then the new zero vanna strike will in general correspond to strike $\hat{k}_{d_2} \neq k_{d_2}$.

In this subsection we will introduce the notation $D\Sigma_{d_2}^{FS}$ for the total change in implied volatility such that

$$\hat{\Sigma}_{d_2}^{FS} = \Sigma_{d_2}^{FS} + D\Sigma_{d_2}^{FS} \tag{4.7}$$

and $d\Sigma_{d_2}^{FS}$ denotes the usual change in the fixed strike implied volatility. Then,

$$dE_t\left[\sigma_{T,T'}\right] \approx D\Sigma_{d_2}^{FS} \tag{4.8}$$

As the total change in implied volatility will involve the usual change plus a change due to movement along the skew in order to make the implied volatility zero vanna again, we can write

$$D\Sigma_{d_2}^{FS} = d\Sigma_{d_2}^{FS} + \frac{\partial \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}} d\ln k_{d_2} + \frac{1}{2}\frac{\partial^2 \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}^2}(d\ln k_{d_2})^2 \tag{4.9}$$

Here $\partial\Sigma_{d_2}^{FS}/\partial \ln k_{d_2}$ is the skew at the initial zero vanna strike, and $\partial^2\Sigma_{d_2}^{FS}/\partial \ln k_{d_2}^2$ denotes the convexity at the initial zero vanna strike, and $d\ln k_{d_2}$ is the change in log-strike from one zero vanna implied volatility to the next.

In order to find $d\ln k_{d_2}$ we make use of equation (4.1). Since $d_2 = 0$ at all times, the change in zero vanna strike can be directly related to the total change in zero vanna implied volatility through equation (4.1):

$$d\ln k_{d_2} = -(T'-T)\left[\Sigma_{d_2}^{FS}\, D\Sigma_{d_2}^{FS} + \frac{1}{2}(D\Sigma_{d_2}^{FS})^2\right] \tag{4.10}$$

from which follows

$$(d\ln k_{d_2})^2 = (\Sigma_{d_2}^{FS}(T'-T))^2(D\Sigma_{d_2}^{FS})^2 \tag{4.11}$$

Let us substitute expressions (4.10) and (4.11) into equation (4.9),

$$\begin{aligned} D\Sigma_{d_2}^{FS} = d\Sigma_{d_2}^{FS} - (T'-T)\frac{\partial \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}}\left[\Sigma_{d_2}^{FS}\, D\Sigma_{d_2}^{FS} + \frac{1}{2}(D\Sigma_{d_2}^{FS})^2\right] \\ + \frac{1}{2}\frac{\partial^2 \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}^2}(\Sigma_{d_2}^{FS}(T'-T))^2(D\Sigma_{d_2}^{FS})^2 \end{aligned} \tag{4.12}$$

After some rearranging,

$$\left(1 + \Sigma^{FS}(T'-T)\frac{\partial \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}}\right) D\Sigma_{d_2}^{FS} = d\Sigma_{d_2}^{FS} + (\cdots)(D\Sigma_{d_2}^{FS})^2 \tag{4.13}$$

where we will now show that in the above equation $(\cdots) \approx 0$.

Consider now a forward starting fixed strike straddle, initially at the zero vanna strike, with a BMS equivalent price denoted by $ST_{d_2}^{BMS}$. Since the BMS equivalent price of the forward starting straddle does not depend on running time t and is also independent of the spot price,

$$dST_{d_2}^{FS,BMS} = 2\nu_{d_2}^{BMS} d\Sigma_{d_2}^{FS} \tag{4.14}$$

where the subscript d_2 means all quantities are evaluated at the zero vanna strike and $\nu_{d_2}^{BMS}$ is the vega of a vanilla option. Under the risk-neutral measure we have

$$E(dST_{d_2}^{FS,BMS}) = 2\nu_{d_2}^{BMS} E(d\Sigma_{d_2}^{FS}) = 0 \tag{4.15}$$

Furthermore, the volatility swap strike must also be a martingale,

$$E(dE_t\left[\sigma_{T,T'}\right]) = 0 \rightarrow E(D\Sigma_{d_2}^{FS}) \approx 0 \tag{4.16}$$

Combining equations (4.15) and (4.16) and taking the expectation of equation (4.13) gives us indeed that $(\cdots) \approx 0$, and hence,

$$\left(1 + \Sigma^{FS}(T' - T)\frac{\partial \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}}\right) D\Sigma_{d_2}^{FS} \approx d\Sigma_{d_2}^{FS} \tag{4.17}$$

What we have done is to express the total required change in implied volatility in order to maintain zero gamma in terms of the usual change in implied volatility which does not preserve zero vanna. As the usual change in fixed strike implied volatility can be traded through a straddle, we have the following self-financing model-free hedge for forward starting volatility swaps:

$$\boxed{dE_t\left[\sigma_{T,T'}\right] \approx D\Sigma_{d_2}^{FS} \approx \frac{dST^{BMS}}{2\nu_{d_2}^{BMS}\left(1 + \Sigma^{FS}(T' - T)\frac{\partial \Sigma_{d_2}^{FS}}{\partial \ln k_{d_2}}\right)}} \tag{4.18}$$

In other words, the change in forward starting volatility swap strike can be approximately hedged, in a self-financing model-independent manner, by dynamically trading zero vanna straddles with a vega weighted and skew adjusted notional.

5 Concluding remarks

We have generalised the model-free approximation for spot starting volatility swaps given in Rolloos and Arslan [2017] to forward starting volatility swaps. Furthermore, we have shown that a self-financing model-free approximate hedge for the forward starting volatility swap can be achieved by dynamically trading zero vanna forward starting straddles with a vega-weighted and skew-adjusted notional. Clearly, from a trading perspective it is far more attractive to trade a limited number of options instead of a continuum of options, which in practice is not feasible to start with.

References

Frido Rolloos and Melih Arslan. Taylor-made volatility swaps. *Wilmott*, 2017.