Trading Autocorrelation

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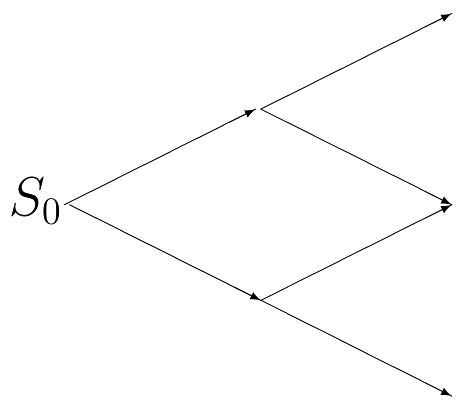
Postscript/PDF files of these overheads can be downloaded from:
`www.petercarr.net` **or**
`www.math.nyu.edu\research\carrp\papers`

Introduction

- Financial observers sometimes classify markets as either *trending* or *reverting*.
- To illustrate the terms, consider a two period binomial model:



- The four paths can be split up into the two trending paths UU and DD and the two reverting paths UD and DU.
- To bet that trending will occur, one could buy an ATM option, delta hedge it initially, and not rebalance.
- To bet instead on the two reverting paths, one can take the opposite positions.
- Alternatively, one can buy or sell ATM straddles (with different transactions costs). This alternative is necessary if the underlying does not trade.

Objectives

- There are three objectives of this presentation which are designed to appeal to:
  1. academics
  2. the buy side/ hedge funds
  3. the sell side/ market makers.
- For academics, we offer definitions for what is meant by the terms ‘trending” and “reverting” in completely general models.
- For investors, we show how to bet on whether markets will be trending or reverting. Our simplest trading strategy uses just futures.
- Finally, for market makers, we show how to price and hedge several path-dependent contracts which should appeal to clients wishing to speculate on whether markets will be trending or reverting. The hedge involves dynamic futures trading and perhaps a static position in one option.
- A distinguishing feature of our analysis is that we never assume anything about the stochastic process governing prices.

Overview

- We show how to synthesize the following new types of path-dependent contracts:
  1. Serial Covariance Swaps
  2. Hyper Options
  3. Overshooters
  4. Upcrossers and Downcrossers.
- All of these contracts have payoffs depending on various measures of autocorrelation.
- All of these contracts can be perfectly hedged without an assumption on dynamics
- Some of these contracts on autocorrelation have important implications for pricing vanilla options when markets are incomplete.
- The relevance of serial correlation in implementing the Black Scholes model under discrete-time trading is explored in Lo and Wang JF 95.

Variance and Covariance Swaps

- Suppose that we partition the time set $[0, T)$ into n time intervals of the form $[t_i, t_{i+1})$, where:
$$0 \equiv t_0 \leq t_1 \leq t_2 \leq ...t_{n-1} \leq t_n \equiv T.$$
- Let F_i denote the futures price at time t_i for maturity T. We assume marking-to-market occurs at each t_i.
- By definition, a variance swap has a payoff at T of:
$$VS_n \equiv \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{F_{i+1} - F_i}{F_i} \right)^2 - K_0^{vs},$$
where K_0^{vs} is chosen so that the variance swap is initially free.
- It is now well known how to price and hedge variance swaps assuming essentially only continuity of the price process (and monitoring and marking-to-market). See Carr & Madan (1998).
- Suppose that the payoff on a covariance swap is defined as:
$$CS_n \equiv \frac{1}{n-1} \sum_{i=1}^{n-1} \left(\frac{F_i - F_{i-1}}{F_{i-1}} \right) \left(\frac{F_{i+1} - F_i}{F_i} \right) - K_0^{cs},$$
where again K_0^{cs} is chosen at time 0 so that the covariance swap has zero cost to enter. In words, the floating part of the payoff is the average of the products of adjacent returns.
- What is the arbitrage-free value of the swap rate K_0^{cs}?

Hedging Covariance Swaps

- Recall that the payoff on a covariance swap was defined as:

$$CS_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right) - K_0^{cs}.$$

- Let $r(t)$ be the deterministic spot interest rate at time t.
- Suppose we do nothing from day 0 to day 1.
- If we hold $\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_i F_{i-1}}$ futures contracts from time t_i to time $t_{i+1}, i = 1, \ldots, n-1$, then we receive $\left(\frac{e^{-\int_{t_{i+1}}^{t_n} r(u)du}(F_i - F_{i-1})}{(n-1)F_i F_{i-1}}\right) \times (F_{i+1} - F_i)$ in marking-to-market proceeds at time t_{i+1}.
- These proceeds grow at the interest rate to be $\left(\frac{(F_i - F_{i-1})}{(n-1)F_i F_{i-1}}\right) \times (F_{i+1} - F_i)$ in marking-to-market proceeds on day t_n.
- Summing over $i = 1, 2, \ldots, n-1$, the sum of the future values of the marking-to-market proceeds by time t_n is: $\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right)$.
- As the initial position is zero futures and the futures trading strategy is trivially self-financing, the arbitrage-free value of the covariance swap rate K_0^{cs} is zero.

Why?

- Recall that the payoff on a covariance swap was defined as:
$$CS_n \equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right) - K_0^{cs}.$$
- It is well known that no arbitrage implies the existence of a probability measure Q equivalent to the original measure P such that the futures price is a martingale.
- This martingale is adapted to the futures price process.
- Hence, payoffs of the form $\int_0^T N_t^f dF_t$ are priced at 0 so long as N_t^f just depends on time and the futures price path up to t.
- The floating part of the covariance swap payoff defined above is just a special case.
- All martingales have increments which are uncorrelated.
- All we have done is to demonstrate the trading strategy in futures that enforces this result.
- Under zero interest rates (eg. Japan), the covariance of returns on any non-dividend paying asset (eg. Nikkei index options) would also be priced at zero.
- One can also trade cross auto-covariance (for zero if written on futures prices).

Trending and Reversion

- Consider a discrete time path of the futures price up to some fixed time $t_n \equiv T$.
- We say that the path is *trending* if:

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right) > 0,$$

  and we say that the path is *reverting* if:

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right) < 0.$$

- Recall that no arbitrage implies the existence of a measure Q such that:

$$E^Q \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right) = 0.$$

- Hence, no arbitrage implies that there must be trending paths with positive Q measure and there must be reverting paths with positive Q measure.
- By the equivalance of Q and P, the same statement holds under the real world probability measure P.

Options on Covariance

- Recall that a fairly priced covariance swap pays off:
$$CS_n = \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{F_i - F_{i-1}}{F_{i-1}}\right)\left(\frac{F_{i+1} - F_i}{F_i}\right).$$
- When the futures price process is unbounded above, the maximum possible loss on a covariance swap is infinite (widows & orphans beware!).
- Even if an investor is prepared to take this risk, it should be recognized by the contract provider that the potential for ex-post default is large if the customer is not well-capitalized.
- Both problems are solved by introducing options on covariance. Suppose a call pays off $(CS_n - K)^+$, while a put pays off $(K - CS_n)^+$. Then the maximum loss to a long position in an option is the initial premium.
- A long call appeals to an investor betting on trending, while a long put appeals to an investor betting on reversion.
- Presumably, under-capitalized investors would not be allowed to sell covariance options unless they put up sufficient collateral.
- Unfortunately, we do not yet know how to create the option payoff without a model, so we will switch gears temporarily.

Hyper Options

- To the pantheon of American, European, Asian, Bermudan, & Russian options, we introduce HYPER options (High Yielding Performance Enhancing Reversible options).
- As usual, a hyper option is issued as either a call or a put.
- A hyper option is similar to an American option in that it can be exercised early, but it also differs from an American option in that it can be exercised an unlimited number of times.
- Exercising a hyper option not only locks in the exercise value, but it also turns a hyper call into a hyper put and vice versa.
- Thus after a hyper call is first exercised, it can be exercised next as a put, then as a call, etc. The strike, maturity, and underlying are never changed.
- Since a hyper option can be exercised an unlimited number of times, all of the exercise proceeds are deferred without interest to maturity.
- As usual, a hyper option need never be exercised, so it has nonnegative value.

Hyper Options on Forward Prices

- In this presentation, we will only consider hyper options written on the forward price F of some underlying asset. We assume that both the hyper option and the forward contract mature at some fixed date T.
- Let K denote the strike price of the hyper option.
- If a hyper call is exercised at any time $t \in [0, T]$, the owner locks in the payoff $F_t - K$, which is received at T.
- Exercising the hyper call converts it into a hyper put with the same underlying, strike, and maturity.
- We do not require that the hyper call be ITM for it to be exercised. If the owner exercises his hyper call while $F < K$ to obtain the ITM hyper put, then $F - K$ is negative so the owner owes $K - F$ to the writer at maturity.
- If a hyper put is exercised at any time $t \in [0, T]$, the owner locks in the payoff $K - F_t$, which is received at T. Exercising the hyper put also reverses it into a hyper call with the same underlying, strike, and maturity.
- At maturity, the hyper option can be exercised for the final time or it can expire worthless.

Get Plenty of Exercise

- We restrict ourselves to exercise strategies which include exercising at maturity if and only if it is ITM.
- We refer to such a strategy as *sensible*. Sensible strategies permit exercise prior to maturity as well.
- We say that a sensible exercise strategy is *optimal* if it is value maximizing.
- Depending on the price path which is realized, we will show that it can be optimal for the owner of a hyper option to exercise early one or more times.
- In fact, at any time prior to maturity, there is always positive probability of multiple optimal early exercises.
- Thus, the writer of a hyper option must find a hedging strategy which defends against these multiple optimal early exercises.
- Ideally, this hedging strategy would also be immune to model risk.

Why Be Hyper?

- Being long a hyper option would appeal to those wishing to speculate on reversion towards the strike.
- While equities typically display low autocorrelation, many commodity prices are thought to mean revert, as do interest rates and volatilities.
- However, it does not follow that anyone wishing to speculate on trending should sell a hyper option naked.
  1. The exercise decisions still reside with the long side, so the the short side might forecast correctly and still lose money.
  2. the maximum upside from writing hyper options naked is just the premium, while the potential loss is unbounded, unless we artificially introduce process restrictions.
- Later, we will consider a modification of a hyper option which addresses the first issue.

The Hyper American

- Recall that a hyper option is a multiply exerciseable American option whose polarity switches on each exercise.
- Since hyper options can potentially be exercised infinitely often, all exercise proceeds are deferred without interest to maturity.
- When the hyper option is written on a forward price as we assume, then at any time there is positive probability of multiple optimal early exercises.
- All of this suggests that a hyper option has greater value than a standard American option on the forward price (which has a positive early exercise premium).

Objects May Appear Larger...

- Assuming only frictionless markets and no arbitrage, we show that a hyper option has exactly the same value as the corresponding European option, regardless of the model.
- Thus, no arbitrage forces the hyper call to have the same value as the European call with the same underlying, strike, and maturity. The analogous statement holds for puts.
- The reason for these surprising results is that all sensible exercise strategies are also optimal.
- Note that this result differs from Merton's classical result for American calls on non-dividend paying stocks. For these options, the optimal exercise strategy is to wait to maturity and exercise if and only if the call is ITM then.

Hedging Hyper Options

- Let C_t^h and P_t^h denote the respective prices at time $t \in [0, T]$ of hyper calls & puts with fixed strike K & fixed maturity T.
- Let C_t^e and P_t^e denote the corresponding European option prices.
- From no arbitrage, put call parity holds for European options:

$$C_t^e - P_t^e = B_t(F_t - K), \qquad t \in [0, T].$$

- Consider the following *polarity matching strategy* for hedging the sale of a hyper option:
  1. If the owner is holding the hyper option as a call, hold a European call in the hedge.
  2. If the owner is holding the hyper option as a put, hold a European put in the hedge.
  3. Finance transitions between European calls and puts using a bond maturing at T.
- This strategy perfectly replicates the payoffs to a hyper option and hence we conclude from no arbitrage that:

$$C_t^h = C_t^e \qquad P_t^h = P_t^e, \qquad t \in [0, T].$$

- It follows that hyper put call parity also holds:

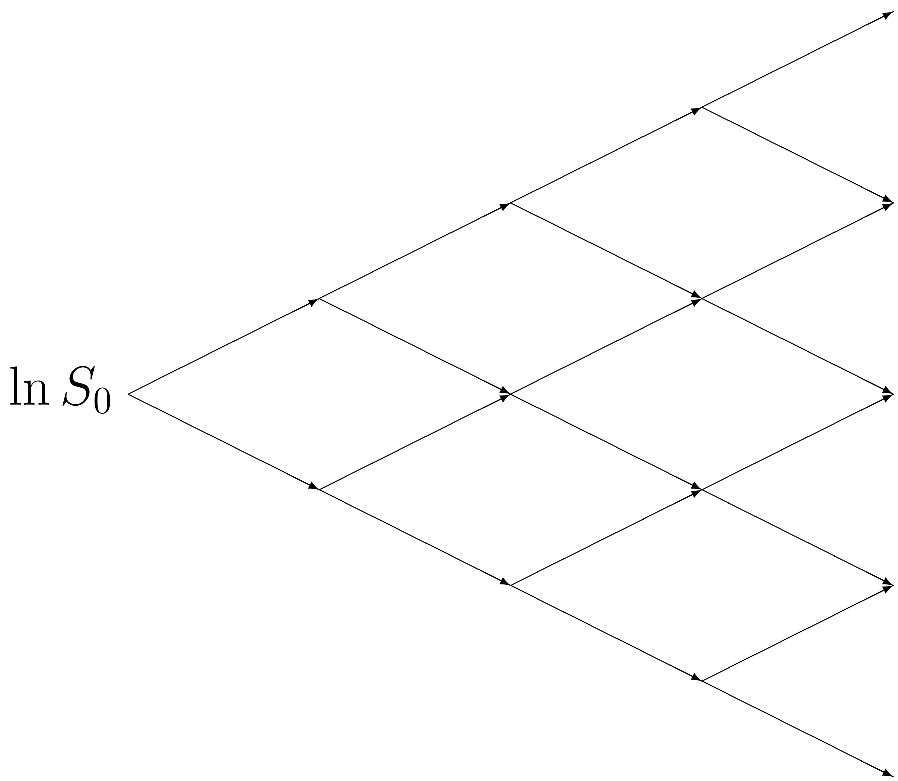
$$C_t^h - P_t^h = B_t(F_t - K), \qquad t \in [0, T].$$

Remarks on the Hyper Option Hedge

- As the polarity matching strategy involves trading European options at every exercise of the hyper option, hedgers worried about transactions costs should note that either polarity can be synthesized using put call parity.
- Hence, one can alternatively hedge a hyper option using a static position in a single European option and a 0-1 dynamic strategy in the underlying similar to the so-called stop-loss start-gain strategy (see Carr Jarrow RFS 90).
- We have extended these results to hyper options on the spot price and to "super options".

Closing Remarks on Hyper Options

- A hyper option gives us a new way of understanding the value of a single European option of the same type.
- For example, consider 2 paths on the following binomial tree:



- Of the paths UUUU and UDUD, which path do you think is more volatile?

Uncertain Volatility

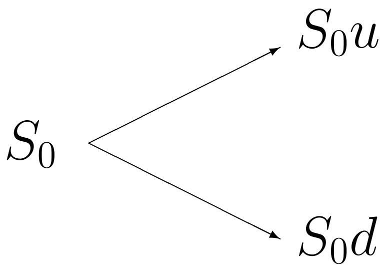
- To a probabilist equating the word "volatility" to quadratic variation of returns, both paths have the same volatility.
- On the other hand, to a statistician who equates volatility to the standard deviation of the terminal log price, the required estimation of the mean implies that the reverting path UDUD has more volatility than the trending path UUUU.
- On the other hand, to an ATM option writer who does not plan to delta-hedge, the trending path UUUU has more volatility than the reverting path UDUD. This writer equates the word "volatility" to the ATM implied to charge initially.
- On the other hand, to an ATM option writer who does plan to delta-hedge, the reverting path UDUD has higher volatility than the trending path UUUU. Again, equating the word volatility to the initial ATM implied, this writer knows that vega and gamma are more negative along the mean-reverting path than the trending path. Forgetting the tree, these greeks become relevant when one is uncertain about the magnitude of squared returns and the possibility of crashes.
- Of course, an ATM option buyer disagrees with the seller on which path is more volatile and an OTM call trader disagrees again.

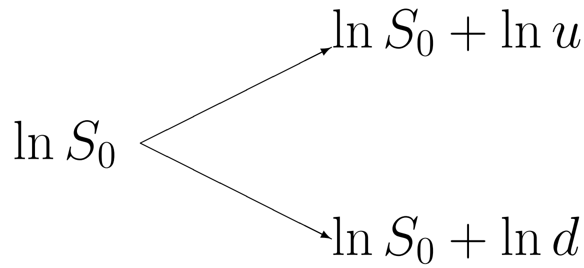
Black Scholes and Beyond

- Much of the analysis on the last page is an attempt to extend the lessons learned in the Black Scholes model to more general settings.
- As our results relating hyper options to European options are model-free, hyper options have much more to say about the value of a European option when markets are incomplete than the Black Scholes model.
- For example, consider an ATM option and suppose that everyone agrees that the reversion of the log price to the log strike increases. Then it is plausible that in an incomplete market, hyper option values would increase.
- From no arbitrage, European option values must then also increase even though the variance of the terminal log price would decrease.
- This illuminates a widely-held fallacy - option prices are not necessarily increasing in (the statistician's definition of) volatility.

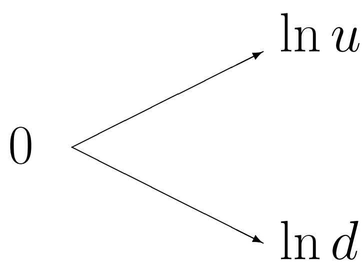
Option Values and Volatility

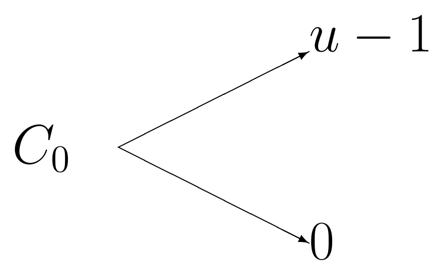
- To illustrate why option prices are not necessarily increasing in volatility in the simplest possible context, consider an ATM call in a single period binomial model under zero interest rates:





where $u > 1 > d > 0$. To further simplify, set $S_0 = K = 1$:





- The risk-neutral probabilities are $q = \frac{1-d}{u-d}$ and $1 - q = \frac{u-1}{u-d}$.
- From elementary probability, the (risk-neutral lognormal) variance is $V_0 \equiv \ln^2\left(\frac{u}{d}\right) q(1-q)$.
- The ATM call value is $C_0 = q(u-1)$.
- Substitution implies $V_0 = \ln^2\left(\frac{u}{d}\right) \frac{(u-1)(1-d)}{(u-d)^2}$ while $C_0 = \frac{(u-1)(1-d)}{u-d}$.

Option Values and Volatility (Con'd)

- Recall that the variance $V_0 = \ln^2\left(\frac{u}{d}\right)\frac{(u-1)(1-d)}{(u-d)^2}$, while the ATM call value $C_0 = \frac{(u-1)(1-d)}{u-d}$.
- Note that $V_0 = \frac{\ln^2(u/d)}{u-d}C_0$.
- For example, suppose $u = \frac{3}{2}$ and $d = \frac{1}{2}$. Then $C_0 = 0.25$ and $V_0 = .3017$.
- Now suppose $u' = 10$ and $d' = \frac{2}{3}$. Then $C_0' = 0.3214$ and $V_0' = .2525$.
- Hence the call value goes up as the variance goes down.
- Note that for the unprimed values, the risk-neutral probabilities are both 1/2. Intuitively, when the risk-neutral probabilities are near 1/2, the variance is large (variance is zero if either risk-neutral probability vanishes). Due to the concavity of the log, the third moment of the return is negative and hence call values are small. In going from the unprimed values to the primed values, we skew the probabilities and the third moment becomes positive. This lowers variance and raises call values.
- The result that option values are not necessarily increasing in volatility is known (see Ravi Jagannathan, JFE, 1984, and Marek Capinski, working paper, 1999).

Option Values and Auto-correlation

- Our model-free result that European options have the same value as hyper options implies that when markets are incomplete, European option premia may be just as much driven by reversion towards the strike as they are by volatility.
- In related work on variance swaps and volatility swaps, we have shown that under a price continuity assumption and deterministic interest rates, delta-hedged *portfolios* of options can deliver realized variance or realized volatility.
- However, when a single option is sold at its initial implied vol σ_{i0}, then the P&L from delta-hedging it continuously at σ_{i0} is:

$$P\&L_T = \int_0^T e^{\int_t^T r(u)du}(\sigma_{i0}^2 - \sigma_t^2)\frac{S_t^2}{2}BS\Gamma(S_t, t; \sigma_{i0})dt,$$

where σ_t is the true instantaneous (random realized lognormal) volatility at time t, and $BS\Gamma$ is Black Scholes gamma.

- Since the $BS\Gamma$ is peaked in S near the strike, paths that hover around the strike amplify the deviation between the initial implied variance rate σ_{i0}^2 and realized variance (i.e. squared returns or the time derivative of quadratic variation).

Option Values & Auto-correlation (Con'd)

- Recall that the P&L from selling an option at its market price and then always delta-hedging it at its initial implied σ_{i0} is:

$$P\&L_T = \int_0^T e^{\int_t^T r(u)du}(\sigma_{i0}^2 - \sigma_t^2)\frac{S_t^2}{2}BS\Gamma(S_t, t; \sigma_{i0})dt.$$

- To the extent that:
  1. it is difficult to hedge or diversify away the randomness in instantaneous volatility, and as a result:
  2. option writers just delta-hedge more or less as described above and they have market power, and:
  3. option writers are risk-averse,

  then the P&L risk will be priced into options. The more the underlying stock price is anticipated to revert about the strike, the more the option writer would charge.

- If in addition, equity option writers are long the stock market, and the leverage effect holds, then this effect is magnified.

- The above expression for terminal P&L can be derived when we realistically allow jumps (See Carr, Lewis, Madan 2000). This expression shows that this reversion effect is magnified further since big down moves tend to be followed by big up moves.

Trading Trending

- Recall that being long a hyper option would appeal to those wishing to speculate on reversion towards the strike.
- However, it does not follow that anyone wishing to speculate on trending should sell a hyper option naked. The short side might forecast trending correctly and still lose money due to adverse exercise decisions by the long side.
- To remedy this deficiency, suppose we contractually restrict the exercise policy to be a *threshold policy*.
- To define this policy, consider 2 time-dependent and possibly discontinuous barriers L_t and H_t with $L_t \leq K \leq H_t$.
- For the threshold policy, all exercises are forced. Furthermore, the 1st forced exercise occurs when F reaches or crosses the boundary on the same side of F_0 as K. Hence, if $F_0 > K$, the new option starts as a put, but if $F_0 < K$, it starts as a call.
- Subsequent to the 1st exercise, the threshold policy forces the owner to exercise each time that an overshoot of K reaches or crosses the relevant barrier. Furthermore at maturity, the option must be exercised if ITM (formally, $L_T = H_T = K$).
- The value of this modified hyper option (called an overshooter) is now the *time value* of the corresponding European option.

An Application to Barrier Options

- Our results on overshooters can be used to get a new simple and general upper bound on the value of barrier options expressed in terms of the values of European options.
- Suppose that the initial forward price is above the strike ($F_0 > K$), so that the overshooter has the same initial value as an OTM European put.
- Further suppose that $L_t = K$ and $H_t = \infty$. Then the OTM European put has the same initial value as a claim paying the first overshoot of K plus the payoff from a DIC with strike and barrier both equal to K.
- Hence, the DIC value is bounded above by the European put value. This generalizes a result in Bowie and Carr 1994 for continuous processes for which the overshoot must be zero.
- As we raise the strike of the DIC to some $K_c > K$, we lower the DIC's value. Hence, the price of the standard put with strike K remains an upper bound:

$$DIC_0(K_c, K) \leq P_0(K), \quad K_c \geq K.$$

Trading Upcrosses

- Consider an interval (K, H) and a time set $[0, T]$.
- An upcross of the interval is completed at the first passage time of the underlying forward price from some level at or below K to some level at or above H. The underlying has to return to K or below to set up for another upcross.
- An upcross is partially completed at maturity if the underlying finishes in (K, H). The fraction of the upcross partially completed at maturity is then $\frac{F_T - K}{H - K}$.
- Consider a claim which at its maturity pays the number of upcrosses, both completed and partial.
- Suppose we restrict the overshooter so that $L_t = K$ and H_t is constant at H. Again the value of this overshooter is just the time value of the corresponding European option.
- If we scale everything by $\frac{1}{H-K}$, then the total (forced) exercise proceeds weakly dominate the number of upcrosses.
- Letting $U_0(K, H)$ be the initial value of the upcrosser, no arbitrage implies:

$$U_0(K, H) \leq \frac{\min[P_0^e(K), C_0^e(K)]}{H - K}.$$

Remarks on Crossers

- Recall that under no arbitrage, the value of a claim paying the number of upcrosses is bounded above:

$$U_0(K,H) \leq \frac{\min[P_0^e(K), C_0^e(K)]}{H-K}.$$

- When no skips over the barriers K and H are allowed (eg. no jumps), then:

$$U_0(K,H) = \frac{\min[P_0^e(K), C_0^e(K)]}{H-K}.$$

- We can just as easily trade downcrossers as well:

$$D_0(L,K) \leq \frac{\min[P_0^e(K), C_0^e(K)]}{K-L}.$$

No skips implies $D_0(L,K) = \frac{\min[P_0^e(K), C_0^e(K)]}{K-L}.$

- All of our results hold under discrete path monitoring as well, although the skipfree condition is more artificial.

Summary and Conclusions

- Serial covariance can be traded using just futures (without a model).
- If the potential for unlimited loss is too scary, one can instead pay a premium and just trade the reversion using long positions in either hyper options, overshooters, or crossers. In this case, loss is limited to the initial premium.
- The definition of volatility is itself volatile.
- In general, European option values are not necessarily increasing in the risk-neutral variance of the terminal log price.
- The model-free result that European options have the same value as hyper options suggests a largely unexplored role for serial covariation in the pricing of European options when markets are incomplete.
- Further research is needed on this "trendy" topic.
- Postscript/PDF files of these overheads can be downloaded from:
  `www.petercarr.net` or
  `www.math.nyu.edu\research\carrp\papers`